UNIVERSITY OF COPENHAGEN
DEPARTMENT OF MATHEMATICAL SCIENCES

Bachelor's project

A Calculation on Cobordism

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Abstract

In this thesis we will study the cobordism ring of oriented manifolds. We introduce the basic definitions of smooth manifolds, and we describe the cobordism equivalence class on oriented and unoriented manifolds. We show the set of equivalence classes of manifolds up to cobordism is a ring with the disjoint union and cartesian product as operations. Given this we give an explicit calculation of the oriented cobordism group in dimension 0,1,2 using the classification of oriented manifolds in these dimensions. For dimension 3 we briefly describe an inductive method by surgery to show every oriented 3-manifold is nulcobordant.

The goal of the rest of the thesis is to classify the oriented cobordism group up to torsion elements in every higher dimension. We introduce vector bundles and show the tautological bundles classify vector bundles. Characteristic classes of vector bundles are introduced, especially the Stiefel-Whitney classes, Chern classes and Pontryagin classes. From this we introduce Pontryagin numbers, and show they are group homomorphisms from the oriented cobordism group to the integers. We calculate the Pontryagin numbers of products of even complex projective spaces, and show there are no linear relations between them in the oriented cobordism ring.

We then describe the Pontryagin-Thom construction and René Thom's proof that the cobordism groups are equal to the homotopy groups of the Thom spectrum. Using this we show the rank of the cobordism group is equal to the rank the subgroup generated by the products of even complex projective spaces, which shows this subgroup contains all non-torsion elements.

As an application of this classification, we prove the Hirzebruch signature theorem by only checking it holds on even complex projective spaces. Lastly as an application of the application, we find a criteria for whether $\#_g \left(S^{4n+2} \times S^{4n+2}\right)$ can have a complex structure on the tangent bundle, by using the Hirzebruch signature theorem.

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1 Introduction

As a general goal of topology, we would like to classify shapes up to a certain structure. In the case of smooth manifolds however the classification up to diffeomorphism is not feasible. Instead based on earlier work of Henri Poincaré, René Thom and Lev Pontryagin defined the cobordism group, where two closed manifolds are cobordant if their disjoint union is a boundary of a compact manifold. It follows that a closed manifold is cobordant to the empty set only if it is a boundary of a compact manifold. Classifying this equivalence relation then gives the answer to the question of whether or not closed manifolds are always boundaries of compact manifolds. The concept of oriented cobordism was similarly defined requiring the manifolds to be oriented, and the boundary orientation to correspond to the orientation of the closed manifolds.
From calculating the cobordism group in lower dimensions, it is clear that this relation is not nearly as strict as diffeomorphism as all closed oriented manifolds of dimension 1 to 3 are the oriented boundary of a manifold. However the cobordism relation still maintain some important invariants of manifolds as signature for example.
Pontryagin showed that the Pontryagin numbers arising from Pontryagin classes of vector bundles, would vanish for oriented nulcobordant manifolds, connecting the theory of characteristic classes and cobordisms. Thom proved that the cobordism groups were isomorphic to the homotopy groups of a certain sequence of spaces called the Thom spectrum. From this Thom could calculate the cobordism ring up to torsion in both the oriented and unoriented case. In 1959 C.T.C Wall computed the cobordism ring completely.
Friedrich Hirzebruch would use this result to relate the signature of topological manifolds to the Pontryagin numbers of smooth manifolds.

This thesis heavily relies on John Milnor's excellent exposition *Characteristic Classes*, whose own contribution to the field can not be overstated.

2 Introduction to Manifolds

We will recall the basic definitions and facts about manifolds.

Definition 2.1 (Topological Manifolds)**.** *A topological manifold M of dimension n is a topological space which is locally homeomorphic to $\mathbb{R}^n$, Hausdorff and second-countable. By locally homeomorphic we mean, there exist for every $x \in M$ an open neighborhood $x \in U$ with a homeomorphism $\phi : U \to \mathbb{R}^n$. Such a map is called a local homeomorphism.*

Definition 2.2 (Topological Manifolds with boundary)**.** *A topological manifold of dimension n with boundary M is defined similarly, except that it is locally homeomorphic to an open subset of $[0, \infty) \times \mathbb{R}^{n-1}$.*
A point $x \in M$ lies in the boundary, denoted ∂M, if there exists a local homeomorphism $\phi : U \to [0, \infty) \times \mathbb{R}^{n-1}$ with $x \in \phi^{-1}(\{0\} \times \mathbb{R}^{n-1})$

Notation 2.3 (Notation for Manifolds)**.** *We will sometimes denote topological manifolds as topological manifolds without boundary, when we want to distinguish them from topological manifolds with boundary. We will also call a manifold of dimension n, a n-manifold. To denote an arbitrary manifold, with or without boundary, we will usually use letters M, N, W. The boundary of a manifold M is denoted by ∂M.*

If $x \in M$ is in the preimage of $\{0\} \times \mathbb{R}^{n-1}$ in one of these local homeomorphism, then it holds for every such homeomorphism[2, p. 29][1]. If there exist a local homeomorphism $\phi : U \to [0, \infty) \times \mathbb{R}^{n-1}$ such that $x \in \phi^{-1}(\{0\} \times \mathbb{R}^{n-1})$, then for every other local homeomorphism $\psi : V \to [0, \infty) \times \mathbb{R}^{n-1}$ with $x \in V$, $x \in \psi^{-1}(\{0\} \times \mathbb{R}^{n-1})$. The boundary of M, ∂M is a topological $(n-1)$-manifold without boundary, While the interior Int M is a topological n-manifold without boundary.

We will restrict our attention to manifolds with a smooth structure. The presence of smooth functions give rise to techniques from Morse theory, and the notion of the tangent bundle which we define later.

Definition 2.4 (Atlas & Smooth Manifolds). *A smooth atlas on a manifold M is a collection of local homeomorphisms from open subsets $\phi : U \to V$ where U is open in M and V open in $\mathbb{R}^n$, called coordinate charts, such that the sources of the coordinate charts covers the whole manifold. The charts have to be smoothly compatible in the sense that for two charts $\phi : U \to V$ and $\psi : U' \to V'$, the transition function $\phi_{U \cap U'} \circ \psi^{-1}_{U \cap U'} : \psi(U \cap U') \to \phi(U \cap U')$ between them is a smooth functions between open subsets of $\mathbb{R}^n$. An atlas is maximal if every local homeomorphism which has smooth transition functions with the elements of the atlas, is included in the atlas.*

A smooth manifold is a topological manifold endowed with a smooth atlas. An atlas on a manifold with boundary is defined similarly, with coordinate charts $\phi : U \to [0, \infty) \times \mathbb{R}^{n-1}$, and transition functions between $\phi : U \to [0, \infty) \times \mathbb{R}^{n-1}$ and $\psi : V \to [0, \infty) \times \mathbb{R}^{n-1}$, $\phi_{U \cap V} \circ \psi^{-1}_{U \cap V}$ being a smooth map.
A map $f : A \to B$ for arbitrary subsets $A \subset \mathbb{R}^n$, is smooth if for every $x \in A$ there exist a neighborhood $x \in U_x$ such that $f_{A \cap U_x}$ can be extended to a smooth map on U_x.

Definition 2.5 (Smooth maps between Smooth Manifolds). *Let M, N be smooth manifolds. A map $F : M \to N$ if for every $x \in M$, we can choose charts ϕ with source $x \in U \subset M$ and ψ with source $f(x) \in U' \subset N$, such that the map $\psi \circ F \circ (\phi_{U \cap F^{-1}(U')})^{-1}$ is a smooth map of open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$.*

From this point on will refer to smooth manifolds as just manifolds.
Another structure that can be added to an manifold is an orientation. For every point $x \in M$ we can choose a coordinate chart around x, $\phi : U \cong \mathbb{R}^n$. Based on this we can calculate $H_n(M, M - x)$ by excision on $(M, U, M - x)$

$$H_n(M, M - x) \cong H_n(U, U - x) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \phi(x)) \cong \mathbb{Z},$$

we can conclude that $H_n(M, M - x)$ has two generators.
We want to be able to make a choice of generator for every $x \in M$. We will denote such a collection of generators by $\mu_x \in H_n(M, M - x)$ for all $x \in M$. Given a compact coordinate ball $\phi : B \to D^n$ with $x, y \in B$, since B is contractible we get

$$H_n(M, M - x) \cong H_n(M, M - B) \cong H_n(M, M - y),$$

given by the inclusion $(M, M - B) \to (M, M - x)$. If a choice of generators send μ_x to μ_y through the above isomorphism for every compact coordinate ball B and $x, y \in B$ then the choice is said to be continuous.

Definition 2.6 (Orientation). *An orientation on a manifold M is a continuous choice of generators $\mu_x \in H_n(M, M - x)$. An oriented manifold is a manifold M together with an orientation on M.*
An orientation on a manifold with boundary M is an orientation on the interior Int M. Similarly an oriented manifold with boundary is a manifold with boundary together with an orientation on it.
If there exist an orientation on a manifold it is orientable.

If a manifold M is smooth, then the orientation for a point $x \in M$ can be given by an orientation on the tangent space[5, p. 122].
We can get an orientation on the tangent space[2] by assigning an ordered basis to each fiber of a manifold M. This pointwise orientation on the whole tangent space is continuous, if every point of M has a local oriented frame. Here a local oriented frame on a manifold is an ordered set of vector

[1] This proof is only for the case of smooth manifolds, introduced below but we will only need it in that case.
[2] The reader can read the definition of the tangent space in the Chapter about Vector Bundles.

fields $(X_1, X_2, ..., X_n)$ on a subset U, such that for each $u \in U$, $(X_{1|u}, X_{2|u}, ..., X_{n|u})$ is an oriented basis. Remember: if we permute an ordered basis, we get another oriented basis if the permutation is even, and reverse oriented if it is odd.

For S^1 we have a global framing given by the vector field, which sends $e^{i\theta} \in S^1$ to $e^{i(\theta+\frac{\pi}{2})} \in T_{e^{i\theta}}S^1$. This determines the orientation on S^1. For S^2 such a global framing does not exist, but there still exist local frames, giving an orientation on S^2.

3 Cobordisms

We will define closed manifolds to be compact manifolds without boundary. We now have the tools to state our main question of this project in a general way: Which closed manifolds are the boundary of compact manifolds?

Definition 3.1 (Closed Manifolds)**.** *A closed manifold M is a compact manifold without boundary.*

By restricting our definitions to compact manifolds, we avoid trivial examples such as any closed manifold M being the boundary of $M \times [0, \infty)$.

Definition 3.2 (Cobordism)**.** *A cobordism is a triple (W, M, N) of W a compact $(n+1)$-manifold and M,N closed n-manifolds, where $\partial W = M \amalg N$. Two manifolds M,N are cobordant if there exist a cobordism (W, M, N).*

Lemma 3.3. *Cobordism gives an equivalence relation on closed manifolds.*

Proof. We let $M \sim N$ if they are cobordant. We will have to show that it is reflexive, symmetric and transitive. We denote the closed unit interval by I.
Since the disjoint union is symmetric, a cobordism (W, M, N) corresponds to (W, N, M), showing that if $M \sim N$ then $N \sim M$. It is reflexive since for any closed manifold M, we have the manifold $M \times I$ with boundary $M \amalg M$, so $M \sim M$. Lastly to show transitivity, if we assume we have cobordisms (W, M, N) and (V, N, D), we can glue W and V along their common boundary N. The proof that gluing gives a well defined smooth manifold up to diffeomorphism can be found here [2, p. 224]. This new manifold gives a cobordism $(W \cup_N V, M, D)$, showing that $M \sim D$, given $M \sim N$ and $N \sim D$. □

Since cobordisms define an equivalence relation, we can take the equivalence classes of closed n-manifolds up to cobordisms. If there is only one equivalence class in this set, then all manifolds are cobordant to the empty manifold, implying that they are boundaries.
We can give the set of closed n-manifolds up to cobordisms an abelian group structure with the disjoint union on representatives. First we have to show it is well defined. If $M \sim M'$ and $N \sim N'$ such that there exist cobordisms (W, M, N) and (V, N, D), then we have the cobordism $(W \amalg V, M \amalg N, M' \amalg N')$, showing that $M \amalg N \sim M' \amalg N'$.
The class $[\emptyset]$ of the empty manifold[3] is the neutral element. Also since we had the cobordism $(M \times I, M \amalg M, \emptyset)$ every class of a manifold is its own inverse element. This is the definition of the cobordism group Ω_n.

We have a commutative bilinear map $\Omega_n \times \Omega_m \to \Omega_{n+m}$ given by the cartesian product on representatives. This is well defined since if we have cobordisms (W, M, M') and (V, N, N') then we have the cobordism

$$((W \times N) \cup_{M' \times N} (M' \times V), M \times N, M' \times N'),$$

showing that products respect equivalence classes.
Since $[(M \amalg N) \times D] = [(M \times D) \amalg (N \times D)]$ and $[M \times N] = [N \times M]$, we get the map is symmetric and bilinear.
We can take the direct sum of all cobordism groups, denoted by Ω_*, and give it a multiplication by the cartesian product on representatives between the summands. The multiplicative identity is the class of the point $[*] \in \Omega_0$.

[3]The empty manifold can be considered a smooth manifold in any dimension, by giving it an empty atlas of dimension n.

Recall that a $\mathbb{Z}_2$-algebra structure on a ring R is a ring-homomorphism $\mathbb{Z}_2 \to R$. We have the unique map $\phi : \mathbb{Z}_2 \to \Omega_*$ sending $[1]$ to $[*]$, since $[*] + [*] = [\emptyset]$. Thus Ω_* is a graded symmetric $\mathbb{Z}_2$-algebra.

3.1 Oriented Cobordisms

We can also make the same constructions for oriented manifolds. An oriented manifold with boundary has an induced canonical orientation on the boundary. The induced orientation is defined such that for every point $x \in \partial M$ with an outward pointing tangent vector v, $(v_1, v_2, ..., v_{n-1})$ is an ordered basis of $T_x\partial M$ with the induced orientation, if and only if $(v, v_1, v_2, ..., v_{n-1})$ is an ordered basis T_xM. In the case of the boundary orientation of a 1-manifold M, $x \in \partial M$ is positively oriented if an outward pointed tangent vector is an oriented basis in M, and negatively oriented if the outward pointed tangent vector gives an unoriented basis.
Based on this we can define an oriented cobordism. Our only difference is, that one of the orientations will be switched, to make sure our definitions work.

Definition 3.4 (Oriented Cobordism)**.** *An oriented cobordism is a triple (W, M, N) of W a compact oriented $(n + 1)$-manifold and M,N closed oriented n-manifolds, where $\partial W = M \amalg -N$, with the induced boundary orientation agreeing with orientation on $M \amalg -N$ given by the orientation of M and $-N$ (here $-N$ is the manifold N with reverse orientation). Two manifolds M,N are oriented cobordant if there exist an oriented cobordism (W, M, N).*

Oriented cobordism also gives an equivalence relation on oriented manifolds, which we will show. Note we have a canonical orientation on a product of oriented manifolds, such that if $(x_1, ..., x_m)$ is an orientend basis for T_xM and $(y_1, ..., y_n)$ is for T_yN then $(x_1, ..., x_m, y_1, ..., y_n)$ is an oriented basis for $T_{(x,y)}M \times N$.
If we take the oriented product $N \times M$ instead, we get a basis $(y_1, ..., y_n, x_1, ..., x_m)$.

Lemma 3.5. *For an oriented m-manifold M and an oriented n-manifold N, the map swapping coordinates $M \times N \to N \times M$ is orientation-preserving if and only if mn is even.*

Proof. Let $(x_1, ..., x_m)$ be an orientend basis for T_xM and $(y_1, ..., y_n)$ be an oriented basis for T_yN. Then we have the oriented basis $(y_1, ..., y_n, x_1, ..., x_m)$ for $T_{y,x}(N \times M)$. By applying the permutation $(1\ 2\ ...\ m + n)^m$ to $(y_1, ..., y_n, x_1, ..., x_m)$, the indices gets shifted to the right m times, giving the ordered basis $(x_1, ..., x_m, y_1, ..., y_n)$. We have

$$(1\ 2\ ...\ m + n) = (1 \quad m + n)(1 \quad m + n - 1)...(1 \quad 2)$$

So $(1\ 2\ ...\ m + n)^m$ consists of $(m + n - 1) \cdot m$ swaps. Therefore it takes $m^2 + mn - m \equiv mn \mod 2$ swaps to get the same basis as $T_{x,y}(M \times N)$. It follows that the map is orientation-preserving if and only if mn is even.

□

We have the standard orientation on I which corresponds to a tangent vector, in the direction of 1.

Lemma 3.6. *Oriented cobordism gives a equivalence relation on oriented closed manifolds.*

Proof. For any oriented manifold M, $I \times M$ has the oriented boundary $M \amalg -M$. This holds since, if $(v_1, ..., v_n)$ is an oriented basis of a point $m \in M$ then $(v, v_1, ..., v_n)$ is an oriented basis of $(0, m) \in 0 \times M$, where v is tangent vector of I going to 1. But since the boundary orientation is given by having an outward pointing vector in the first coordinate, the orientation has to be swapped on M to get an oriented basis on $0 \times M$. However on $1 \times M$, outward pointing vectors are oriented at $1 \in I$, such that the boundary orientation will agree with the original orientation of M. Thus we have the reflexive property from $(I \times M, M, M)$.
If we have an oriented cobordism (W, M, N) then $(-W, N, M)$ will also be an oriented cobordism, since the boundary orientation will be swapped, giving the symmetric property.

Given two oriented cobordisms (W, M, N), (V, N, D), W and V can be attached along N. The gluing will happen along an orientation reversing map, which ensures that $W \cup_N V$ is an oriented manifold, such that the inclusions of W, V into $W \cup_N V$ are orientation-preserving. This gives an oriented cobordism $(W \cup_N V, M, D)$ showing transitivity, and such that it is an equivalence relation.
□

As before we can take the set of equivalence classes of oriented n-manifolds with the disjoint union as group operation: This is the oriented cobordism group Ω_n^+, with the inverse of a manifold given by the same manifold with reverse orientation, and neutral element $[\emptyset]$ with the unique orientation. As in the unoriented case, Ω_*^+ is a graded ring with the cartesian product, with identity element $[*]$ with the positive orientation. From our above considerations the ring is graded commutative, meaning $[N \times M] = [(-1)^{nm} M \times N]$.

3.2 Calculation of Oriented Cobordism groups

Since the definition of the cobordism groups depends on all closed manifolds, or all oriented closed manifolds in the oriented case, they are not easy to calculate. In dimension 0,1 and 2, however we have simple classifications up to (orientation-preserving) diffeomorphism of (oriented) closed manifolds, which is a finer equivalence class than (oriented) cobordism.

Lemma 3.7. *If two (oriented) manifolds are (orientation-preserving) diffeomorphic, then they are (oriented) cobordant*

Proof. If M and N are oriented diffeomorphic, then glue the manifolds $I \times M$ and $I \times N$ along $\{1\} \times M$ and $\{0\} \times N$ by the diffeomorphism. Since $\{1\} \times M$ has the orientation $-M$ they are glued along an orientation reversing diffeomorphism, which make it so $(I \times M) \cup_{-M \sim N} (I \times N)$ has the orientation of $(I \times M)$ and $(I \times N)$ on each of its parts. The boundary of this is $M \amalg -N$, showing M and N are oriented cobordant.

The same proof works in the unoriented case, when we take our constructions to be unoriented. □

Given this we can calculate the groups in these 3 cases. For Ω_0^+ the closed manifolds are just a finite amount of points each attached a sign which is the orientation. The sums of these signs are invariant under cobordism, since any cobordism between 0-manifolds consists of line segments between an equal amount of positive and negative points, since the boundary orientation on I gives the negative orientation at 0 and positive at 1.

Definition 3.8 (Total sign)**.** *The total sign of an oriented 0 manifold is the integer $a - b$, where a is the number of positive oriented points, and b is the amount of negative oriented points.*

It follows that $M \sim N$ if and only if $M \amalg -N \sim \emptyset$, then the total sign of $M \amalg -N$ is 0, meaning that M and N have the same total sign. Therefore $\Omega_0^+ = \mathbb{Z}$, by sending a class of oriented points to its total sign.

Ω_1^+ and Ω_2^+ are both easier, since the only closed connected orientable manifolds are circles and genus-g surfaces, which are both boundaries of D^2 and a genus-g handlebody respectively, showing both groups are 0.

3.2.1 Calculation of Ω_3^+

In dimension 3 we will choose a different strategy for finding the cobordism group, than the previous examples. Rather given an arbitrary closed oriented 3-manifold, we will describe an algorithm to alter it by surgery without changing its cobordism class. This way we can simplify any manifold into the 3-sphere, which is a boundary, showing that $\Omega_3^+ = 0$.

To do this, first we need a way to write our manifolds in a more systematic way. Even though we don't have a classification available, we have the next best thing, a decomposition. Through techniques of Morse Theory, we can decompose any closed orientable 3-manifold into two genus-g handlebodies glued together. Such a decomposition is called a Heegaard diagram[3, p. 181].

Given this by performing surgery along a specific curve, we get a cobordism with a manifold consisting of two genus-$(g-1)$ handlebodies[6]. Using this inductively we get a cobordism to a union of two genus-0 handlebodies which is just S^3, which is nul-cobordant. It follows that $\Omega_3^+ = 0$.

Even though we have been able to calculate the first oriented cobordism groups, we would like to be able to describe the structure more for all dimensions. But for this to possible we need to develop some machinery.

4 Vector Bundles

Our study of cobordisms leads us to vector bundles of smooth manifolds. We will primarily work with vector bundles with 3 different kinds of structures: oriented, unoriented and complex vector bundles.

Definition 4.1 (Real Vector Bundles). *A real n-vector bundle ξ is a tuple (B, E, π) consisting of a base space B, a total space E and a surjection $\pi : E \to B$ such that the fibers have a real vector space structure of dimension n, and for each point $b \in B$ there exist a neighborhood $b \in U \subseteq B$ with local trivializations $t : U \times \mathbb{R}^n \to \pi^{-1}(U)$, which is, when restricted to a point in U, a vector space isomorphism.*

Definition 4.2 (Complex Vector Bundles). *A complex n-vector bundle ω is a tuple (B, E, π) consisting of a base space B, a total space E and a surjection $\pi : E \to B$ such that the fibers have a complex vector space structure of dimension n, and for each point $b \in B$ there exist a neighborhood $b \in U \subseteq B$ with local trivializations $t : U \times \mathbb{C}^n \to \pi^{-1}(U)$, which is, when restricted to a point in U, a complex vector space isomorphism.*

Definition 4.3 (Oriented Vector Bundles). *An oriented n-vector bundle ω is a real vector bundle, where the fibers are oriented vector fields and the local trivializations $t : U \times \mathbb{R}^n \to \pi^{-1}(U)$ are oriented vector space isomorphism when restricted to a fiber.*

Notation 4.4. *We will usually denote arbitrary real bundles, unoriented or oriented, by ξ or η. The total space will be denoted $E(\xi)$, the base space by $B(\xi)$ and the surjection by π or π_ξ unless we state otherwise. If the given bundle is clear, we will refer to the spaces as E, B.*

For complex bundles we will use ω with total space $E(\omega)$, base space $B(\omega)$ and surjection π.

4.1 Examples of vector bundles

The simplest example of a vector bundle is the trivial n-bundle ϵ^n, $\mathbb{R}^n \times B$ which is globally trivial. All bundles look like this bundle locally from our definition, but their global structure can vary. A vector bundle is trivial, if and only if there exist n cross-sections that are linearly independent in each fiber[5, p. 18]. There also exist an oriented trivial n-bundle and a complex trivial n-bundle.

The tangent bundle is another important example defined for all smooth manifolds.

Definition 4.5 (Tangent Bundle). *The tangent bundle on a n-manifold M associates to each point in M the space of the tangents vectors, which is a n-dimensional vector space.*

For the tangent bundle on a n-manifold M, we use both the notation TM, τ_M or τ^n. There exist local trivilizations $\pi : \mathbb{R}^n \times U \to \pi^{-1}(U)$ based on charts of the manifold $\phi : \mathbb{R}^n \to U$. The tangent bundle on S^{2n} gives us our first example of a non-trivial bundle, since the hairy ball theorem shows every cross section will be 0 somewhere. Smooth manifolds will be called parallelizable if their tangent bundle is trivial. A manifold together with a complex structure on the tangent bundle is an almost complex manifold. $\mathbb{C}^n$ has a canonical complex structure on the tangent bundle by the canonical homeomorphism $T_c\mathbb{C}^n \cong \mathbb{C}^n$.

Definition 4.6 (Complex Manifold). *A $2n$-manifold M with a complex structure on the tangent bundle is a complex n-manifold, if it has an atlas of homeomorphisms from open subsets $U \in M$ to $V \in \mathbb{C}^n$, such that the map on tangent spaces of these homeomorphisms are complex linear.*

An important class of complex manifolds are the complex projective spaces $\mathbb{C}\mathrm{P}^n$.
Another class of bundles are the tautological bundles on Grassmanians, which have an unoriented, oriented and complex version.

Definition 4.7 (Oriented Tautological Bundles). *The unoriented tautological bundle γ_k^n is a real n-bundle on the unoriented real Grassmanian $G_n(\mathbb{R}^{n+k})$. It consists of pairs (x, v) where x is a n-plane in $\mathbb{R}^{n+k}$ and v is a vector in x.*

Definition 4.8 (Unoriented Tautological Bundles). *The unoriented tautological bundle $\tilde{\gamma}_k^n$ is an oriented n-bundle on the oriented real Grassmanian $\tilde{G}_n(\mathbb{R}^{n+k})$. It consists of pairs (x, v) where x is an oriented n-plane in $\mathbb{R}^{n+k}$ and v is a vector in x.*

Definition 4.9 (Complex Tautological Bundles)**.** *The complex tautological bundle γ_k^n is a complex n-bundle on the complex Grassmanian $G_n(\mathbb{C}^{n+k})$. It consists of pairs (x, v) where x is a complex n-plane in $\mathbb{C}^{n+k}$ and v is a vector in x.*

Even though the complex and unoriented tautological bundle are denoted the same, the distinction will be made clear from context.
In the unoriented case γ_1^1 we get the "simplest" non-trivial vector bundle, also called the Möbius bundle, as the total space is a Möbius band while the base is $\mathbb{RP}^1 \cong S^1$.

4.2 Maps between bundles

We can define a map between vector bundles ξ, η as a pair of maps $g : E(\xi) \to E(\eta)$, $f : B(\xi) \to B(\eta)$ which satisfy the commutative diagram

$$\begin{tikzcd} E(\xi) \arrow[r, "g"] \arrow[d] & E(\eta) \arrow[d] \\ B(\xi) \arrow[r, "f"] & B(\eta), \end{tikzcd} \tag{1}$$

where we require g to restrict to linear isomorphisms on the fibers of the projection map. If we have a bundle map on the form (g, Id_B), then we will say the two bundles are isomorphic, and this will be the equivalence relation we will study vector bundles up to.
We will denote the set of isomorphism classes of real n-vector bundles of M by $\mathcal{B}_n(M)$. For oriented and complex they will be denoted $\mathcal{B}_n^+(M)$ $\mathcal{B}_n^{\mathbb{C}}(M)$ respectively.
If ξ, η are bundles over a space M with $E(\xi) \subset E(\eta)$, then ξ is a subbundle of η. If we instead have two bundles ξ, η defined over the same space M, with a map $g : E(\xi) \to E(\eta)$ satisfying the diagram

$$\begin{tikzcd} E(\xi) \arrow[rr, "g"] \arrow[dr] & & E(\eta) \arrow[dl] \\ & M, & \end{tikzcd}$$

with g being linear and injective on fibers, then ξ is isomorphic to a subbundle of η. We will usually describe the bundle map, by describing the map on fibers. If this description depends on local trivializations, we have to ensure it is independent of choice of local trivialization.

4.3 Operations on vector bundles

The vectors bundles we will work with in this project, will all be constructed from the tangent bundle, trivial bundle and the tautological bundle. From these 3 bundles we can generate other bundles of interest. First we have the product bundle of the cartesian product of base spaces.

Definition 4.10 (Product Bundle)**.** *Given two unoriented bundles ξ with projection π_1, and η with projection π_2, the product bundle $\xi \times \eta$ is the bundle $(B(\xi) \times B(\eta), E(\xi) \times E(\eta), \pi_1 \times \pi_2)$, with the fibers having the vector space structure of the product of the fibers of ξ and η.*
If the bundles are oriented, the product fiber has the product orientation, and if it is complex, the product fiber has product structure of complex vector spaces.

We also have the pullback bundle.

Definition 4.11 (Pullback Bundle)**.** *Given a bundle ξ and a map $f : A \to B(\xi)$ we have the pullback.*

$$\begin{tikzcd} E(f^*(\xi)) \arrow[r] \arrow[d, "\pi_f"] & E(\xi) \arrow[d, "\pi"] \\ A \arrow[r, "f"] & B(\xi). \end{tikzcd}$$

Using this we can define the pullback bundle $f^*(\xi)$ as the tuple $(A, E(f^*(\xi)), \pi_f)$.

Remember the pullback in topology is the subspace of $A \times E(\xi)$ consisting of pairs (a, v) such that $f(a) = \pi(v)$, together with subspace topology of $A \times E(\xi)$. The fiber $\pi_f^{-1}(a)$ will then be canonically identified with the fiber of the image $\pi^{-1}(f(a))$ and will be given the same vector space structure.

Given two vector bundles ξ, ϵ on the same base space B, we can combine the product bundle with the pullback bundle on the diagonal map $\Delta : B \to B \times B$ to get the Whitney sum $\xi \oplus \epsilon$ of the two bundles.

Another example of a pullback bundle is the restriction of a bundle ξ to $\xi_{|A}$, given by an inclusion $A \hookrightarrow B(\xi)$.
We have the following lemma connecting the pullback bundle to bundle maps.

Lemma 4.12. *There exists a bundle map (g, f) from η to ξ if and only if η is isomorphic to $f^*(\xi)$*

Proof. The diagram of the pullback bundle gives a bundle map (f^*, f) from $f^*(\xi)$ too ξ. Any bundle that is isomorphic to $f^*(\xi)$ also has a bundle map by composing the isomorphism with (f^*, f).
On the other hand if we have a bundle map $(h, f) : \eta \to \xi$, then we get a commutative diagram from the universal property of pullbacks.

$$\begin{tikzcd}
E(\eta) \arrow[drr, bend left, "h"] \arrow[dr, dashed, "\exists!\phi"] \arrow[ddr, bend right] & & \\
& E(f^*(\xi)) \arrow[r, "f^*"] \arrow[d] & E(\xi) \arrow[d] \\
& B(\eta) \arrow[r, "f"] & B(\xi).
\end{tikzcd}$$

The pair $(\phi, Id_{B(\eta)})$ is a bundle map from η to $f^*(\xi)$, since on fibers $x \in B(\eta)$, $\phi_x = (f_x^*)^{-1} \circ h_x$ it is a composition of linear isomorphims. It follows η and $f^*(\xi)$ are isomorphic bundles. $\square$

Lastly we have the orthonormal bundle, but to define this we need an inner product on fibers. The following definition gives us this.

Definition 4.13 (Euclidean Bundle)**.** *For a bundle ξ we have the pullback $E(\xi) \times_B E(\xi)$ of the diagram*

$$\begin{tikzcd}
E(\xi) \times_B E(\xi) \arrow[r] \arrow[d] & E(\xi) \arrow[d, "\pi"] \\
E(\xi) \arrow[r, "\pi"] & B.
\end{tikzcd}$$

If a continous map $q : E(\xi) \times_B E(\xi) \to \mathbb{R}$ is an inner product when restricted to a fiber $F_b \times F_b$, then q is an euclidean structure on ξ. A bundle ξ together with an eculidean structure is an euclidean bundle.

A trivial bundle with a canonical trivialization has a canonical euclidean structure, given by the usual inner product on each fiber.

Lemma 4.14. *Every bundle ξ over a paracompact space has an euclidean structure.*

Proof. Let $(U_i)_{i \in I}$ be an open cover of ξ, such that the bundle is trivial over each U_i. Then since the base space B is paracompact, there exists a partitions of unity $(\psi_i)_{i \in I}$ of (U_i). Since the bundle is trivial over each U_i, we can choose inner products $q_i : E(\xi)_{|U_i} \times E(\xi)_{|U_i} \to \mathbb{R}$. Then

$$q(e, e') = \sum_{i \in I} \psi_i(\pi(e)) \cdot q_i(e, e')$$

is a inner product on ξ. First note it is well defined, by letting $\psi_i \cdot q_i$ be equal to 0 outside of U_i since sup $\psi_i \subset U_i$. Secondly it is symmetric bilinear since it is a sum of symmetric bilinear functions when restricted to fibers. Lastly it is positive definite, since for any $v \in E(\xi)$ we can chooes ψ_i with $\psi_i(\pi(v)) > 0$. Then since q_i positive definite

$$0 < \psi_i(\pi(v)) \cdot q_i(v, v) \leq q(v, v)$$

Showing q is a inner product on ξ. $\square$

We can now define the normal bundle.

Definition 4.15 (Normal Bundle)**.** *Given an inclusion $\xi \subset \eta$ of a n-bundle inside an euclidean m-bundle, the normal bundle $\xi^{\perp}$ is a $(m-n)$-bundle, with fibers given by the orthogonal space of the fibers of ξ inside the fibers of η.*

The proof that this gives a subbundle, can be found here[2, p. 267]. The isomorphism type of the normal bundle, does not depend on the choice of euclidean bundle.

The Whitney sum of a subbundle with its normal bundle is canonically isomorphic to the total bundle[5, p. 28]

If we have an embedding of a manifold $i : M \to N$ then we have an inclusion of bundles $TM \hookrightarrow TN_{|M}$. If $TN_{|M}$ has an euclidean structure, then we have the normal bundle ν_M of the embedding. The tubular neighborhood theorem states there is a diffeomorphism ϕ from the total space of ν_M to an open neighborhood of $i(M)$ such that:
1) If i' is the zero-section of M in $E(\nu_M)$ then $\phi \circ i' = i$.
2) Given the identifications $\alpha : (TE(\nu_M))_{|M} \xrightarrow{\sim} TM \oplus \nu_M$ and $\beta : TN_{|M} \xrightarrow{\sim} TM \oplus \nu_M$ we get $T\phi = \beta^{-1} \circ \alpha$.
The proof of the tubular neighborhood theorem can be found here[5, p. 115].

4.4 Relations between different types of bundles

We will here describe some relations between unoriented, oriented and complex bundles. There are forgetful functors from oriented and complex bundles to unoriented bundles, considering complex n-bundles to be real $2n$-bundles. Moreover all complex bundles have a canonical orientation.

Lemma 4.16. *Any complex bundle has a canonical orientation.*

Proof. Indeed if we choose a complex basis over a fiber $(z_1, z_2, ..., z_n)$ we have the corresponding real basis $(z_1, z_1 i, z_2, z_2 i, ..., z_n, z_n i)$. The orientation does not depend of our choice of basis, since any permutation of our basis, will result in an even permutation of $(z_1, z_1 i, z_2, z_2 i, ..., z_n, z_n i)$ since for every pair of complex element swapped, we get two pairs of real elements swapped. □

Based on this, we also have a forgetful functor to the oriented bundles, by forgetting the complex structure, but maintaining the canonical orientation.

So far we have only talked about forgetting structure, but we can also get a complex bundle from every bundle by complexification. The complexification of a real bundle ξ is a complex bundle $\xi \otimes \mathbb{C}$, with fibers $\xi_b \otimes_{\mathbb{R}} \mathbb{C}$. The complex structure of a complex vector bundle ω can be described by a $\mathbb{R}$-linear map on fibers $J : \omega_b \to \omega_b$ with $J \circ J(x) = -x$, which corresponds to scalar multiplication by i. For $\xi \otimes \mathbb{C}$ elements of the fiber can be written as (x, y) with the map $J(x, y) = (-y, x)$ or $x \otimes c$ with $J(x \otimes c) = x \otimes ic$

If a real isomorphism ϕ of complex bundles upholds $\phi(J(x)) = J(\phi(x))$ then it is an isomorphism of complex bundles. The conjugate bundle $\overline{\omega}$ of a complex bundle ω, is the same bundle with the complex structure given by $\overline{J} = -J$.

Lemma 4.17. *The bundle $\overline{\xi \otimes \mathbb{C}}$ is canonically isomorphic to $\xi \otimes \mathbb{C}$.*

Proof. We have the bundle map f from $\xi \otimes \mathbb{C}$ to $\overline{\xi \otimes \mathbb{C}}$ given on fibers by sending $x \otimes c$ to $x \otimes \overline{c}$. This is a real bundle map since it is canonically defined, so we just need to show it respects the complex structure, but we have

$$f(J(x \otimes c)) = f(x \otimes ic) = (x \otimes -i\overline{c}) = \overline{J}(x \otimes \overline{c}) = \overline{J}(f(x \otimes c))$$

So the map f is a complex bundle map and is such an isomorphism. □

If our bundle already has an orientation or complex structure, the complexification will contain this structure, in a sense described by the two lemma below.

Lemma 4.18. *For a complex bundle ω, $\omega \otimes \mathbb{C}$ is canonically $\omega \oplus \hat{\omega}$, where $\hat{\omega}$ is the complex conjugate bundle.*

Proof. For the bundle $\omega \otimes \mathbb{C}$ the complex structure is given by $J(x, y) = (-y, x)$.
Meanwhile the complex structure on $\omega_b \oplus \bar{\omega}_b$ is given by $J(x, y) = (ix, -iy)$. We can see from this, that they are not obviously isomorphic.

We can however construct a map $g : \omega_b \to \omega_b \otimes \mathbb{C}$ by $g(a) = (a, -ia)$. We have

$$g(J(a)) = g(ia) = (ia, a) = J(a, -ia) = J(g(a)),$$

so the map is complex linear. We also define a map $h : \hat{\omega}_b \to \omega_b \otimes \mathbb{C}$ by $h(a) = (a, ia)$, which is also complex linear since

$$h(J(a)) = h(-ia) = (-ia, a) = J(a, ia) = J(h(a)).$$

Based on this we can define a map $g \oplus h : \omega_b \oplus \hat{\omega}_b \to \omega_b \otimes \mathbb{C}$, which is surjective since

$$(a, b) = g\left(\frac{a + ib}{2}\right) + h\left(\frac{a - ib}{2}\right),$$

which gives an canonical isomorphism of fibers not depending on trivializations, which therefore extend to a bundle map. ☐

Lemma 4.19. *For an oriented bundle ξ, $(\xi \otimes \mathbb{C})_{\mathbb{R}}$ is isomorphic to $\xi \oplus \xi$ under an orientation preserving map if $\frac{n(n-1)}{2}$ is even, and orientation reversing if it is odd.*

Proof. Based on an ordered basis $(v_1, v_2, ..., v_n)$ of ξ_b the orientation of $\xi_b \otimes \mathbb{C}$ is given by

$$(v_1, v_1 i, v_2, v_2 i, ..., v_n, v_n i).$$

On the other hand if $(v_1, v_2, ..., v_n)$ induce the orientation on ξ_b then the orientation of $\xi_b \oplus i\xi_b$ is given by $(v_1, v_2, ..., v_n, v_1 i, v_2 i, ..., v_n i)$.

If we shift $v_1 i$ to the left in the second basis $(n-1)$ times, it will be in same position as in the first basis. It will take $v_2 i$ $(n-2)$ shifts to the left to get it to the correct position, and so on until the two basis have the same order. From this we have swapped the orientation $\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$ times, so the isomorphism preserves orientation if and only if $\frac{n(n-1)}{2}$ is even. ☐

These result will be useful later as we will construct characteristic classes from the complexification of a bundle.

5 The Universality of the tautological bundle

We will now show why the tautological bundles are important. This result has a version for the oriented, unoriented and complex Grassmannian, but for the purpose of variation we will describe the complex case.
First however we need to define the infinite Grassmannian. Notice the inclusion $\mathbb{C}^{n+k} \hookrightarrow \mathbb{C}^{n+k+1}$, into the first $n + k$ coordinates, induces an inclusion $G_n(\mathbb{C}^{n+k}) \hookrightarrow G_n(\mathbb{C}^{n+k+1})$.

Definition 5.1 (Infinite Grassmannian)**.** *The complex infinite Grassmannian $G_n(\mathbb{C}^\infty)$ is the direct limit of the inclusions $G_n(\mathbb{C}^{n+k}) \hookrightarrow G_n(\mathbb{C}^{n+k+1})$. The unoriented infinite Grassmannian comes form the inclusions $G_n(\mathbb{R}^{n+k}) \hookrightarrow G_n(\mathbb{R}^{n+k+1})$, and the oriented comes from $\tilde{G}_n(\mathbb{R}^{n+k}) \hookrightarrow \tilde{G}_n(\mathbb{R}^{n+k+1})$.*

Here we mean by $\mathbb{C}^\infty$ the direct sum of $\mathbb{N}$ copies of $\mathbb{C}$, such that a n-plane in $\mathbb{C}^\infty$ lies in $\mathbb{C}^m$ for m large enough.

We also have a tautological bundle γ^n which is defined as in the finite case.

Definition 5.2 (Tautological Bundles over infinite Grassmannians)**.** *The complex tautological bundle γ^n is a complex n-bundle on the complex Grassmanian $G_n(\mathbb{C}^\infty)$, consists of pairs (x, v) where x is an complex n-plane in $\mathbb{C}^\infty$ and v is a vector in x.*

We denote $[A, B]$ to be the set of equivalence classes of maps $f : A \to B$ up to homotopy.

Theorem 5.3. *For any manifold M, there is a bijection between $\mathcal{B}_n^{\mathbb{C}}(M)$ and $[M, G_n(\mathbb{C}^\infty)]$. More explicitly the map ϕ assigning to every homotopy class of functions $f : M \to G_n(\mathbb{C}^\infty)$ the isomorphism class $[f^*(\gamma^n)]$, is a bijection.*

We will split the proof in two lemmas.

Lemma 5.4. *for every complex n-bundle ω over a paracompact space there exist a bundle map into γ^n.*

Proof. We will show the existence by constructing a map $\hat{f} : E(\omega) \to \mathbb{C}^\infty$. which is linear and injective on the fibers of ω .
From this we can construct a bundle map, since for $e \in E(\omega)$, $\hat{f}(\pi^{-1}(\pi(e)))$ is a n-plane from the assumption, so we can define the bundle map $f(e) = (\hat{f}(\pi^{-1}(\pi(e))), \hat{f}(e))$.

We can choose a countable open cover (U_m) of $B(\omega)$ such that the restrictions $\omega_{|U_m}$ are trivial.[5, p. 66]. We choose a partition of unity (ψ_m) of (U_m) [2, p. 43]. Since the bundle is trivial over these subsets, we can define maps h_m by $\pi^{-1}(U_m) \xrightarrow{\sim} U_m \times \mathbb{C}^n \xrightarrow{p} \mathbb{C}^n$, where the first map is a trivilization of (U_m). Combining this with our partition, we get a map

$$h'_m(e) = \begin{cases} \psi_m(e) \cdot h_m(e) & e \in \pi^{-1}(U_m) \\ 0 & else \end{cases}$$

We can now define the map $\hat{f} : E(\xi) \to (\mathbb{C}^n)^\infty \cong \mathbb{C}^\infty$ by $\hat{f}_m(e) = h'_m(e)$. Each h'_m is linear on fibers, and for each fiber there exist h'_m which is injective. Lastly only finitely many of the maps are non-zero over a point, ensuring that each fiber is mapped to a plane in C^a for a large enough. This shows the existence of $\hat{f}$ finishing the proof. ☐

For the next lemma we will need the notion of a bundle homotopy

Definition 5.5 (Bundle homotopy)**.** *If we have two bundle maps (a, b) and (c, d) from ξ to η then a bundle homotopy between them is a map*

$$\begin{array}{ccc} E(\xi) \times I & \xrightarrow{h} & E(\eta) \\ \downarrow & & \downarrow \\ B(\xi) \times I & \xrightarrow{g} & B(\eta), \end{array}$$

such that $(h(-, t), g(-, t))$ is a bundle map for each $t \in I$ and $(h(-, 0), g(-, 0)) = (a, b)$ and $(h(-, 1), g(-, 1)) = (c, d)$

Lemma 5.6. *Two bundle maps from the same bundle to the tautological bundle are bundle-homotopic.*

Proof. Let f, g be two bundle maps from a bundle ω to γ^n. We use again that a bundle map determines a map $\hat{f} : E(\omega) \to \mathbb{C}^\infty$ linear and injective on fibers. If $\hat{f}$ and $\hat{g}$ are never negative multiples of each other, then the map $\hat{h} : E(\omega) \times I \to \mathbb{C}^\infty$ given by

$$\hat{h}(e, t) = t \cdot \hat{f}(e) + (1 - t) \cdot \hat{g}(e),$$

is for every $t \in I$, injective on fibers, since the vectors are never negative multiples of each other. This gives a bundle homotopy $h(e, i) = (\hat{h}_i(\pi^{-1}(\pi(e))), \hat{h}_i(e))$.

We can reduce the general case to this special case, by modifying our maps. First we define two bundle maps $d_1, d_2 : \gamma^n \to \gamma^n$. These are also induced by maps $\hat{d_1}, \hat{d_2} : \mathbb{C}^\infty \to \mathbb{C}^\infty$ where $\hat{d_1}$ sends the i-th coordinate to $(2i - 1)$-th and $\hat{d_2}$ sending it to $2i$-th and leaving the rest 0.
If $\hat{f}(e) \neq 0$ assume the last non-zero coordinate is in index m. Then the last non-zero coordinate in $\hat{d_1} \circ \hat{f}(e)$ is in the $2m - 1$-th index, and in the $2m$-th index for $\hat{d_2} \circ \hat{f}(e)$, so the vectors cannot be negative multiples of each other. Also for any bundle maps f and g, the bundle maps $\hat{d_1} \circ \hat{f}(e)$ and $\hat{d_2} \circ \hat{g}(e)$ are not negative multiples of each other since, one is only non-zero in odd indexes and the other only in even indexes. We can now use our first part to conclude that

$$f \sim d_1 \circ f \sim d_2 \circ g \sim g.$$

so f and g are bundle homotopic. ☐

Proof of Theorem 5.3. The fact that every complex n-bundle has an unique bundle map into the tautological bundle up to homotopy, let us define a map $\psi : \mathcal{B}_n^{\mathbb{C}}(M) \to [M, G_n(\mathbb{C}^\infty)]$ which sends a bundle ω to the homotopy class of the map of base spaces of a bundle map $\omega \to \gamma^n$.
The map has an inverse sending a map to the pullback bundle of the map. Indeed this is well defined since homotopic maps leads isomorphic bundles[4, p. 184].
Since the pullback bundle comes attached with a bundle map

$$\begin{array}{ccc} E(f^*\gamma^n) & \longrightarrow & E(\gamma^n) \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & G_n(\mathbb{C}^\infty), \end{array} \tag{2}$$

we get the same map on base space we started with, which shows $\psi \circ \phi = Id_{[M, G_n(\mathbb{C}^\infty)]}$. For the other direction, since a bundle ω is isomorphic to the pullback bundle $f^*(\gamma^n)$ for some f, if there exists a bundle map $(g, f) : \omega \to \gamma^n$ it will get send to the same isomorphism class, showing $\psi \circ \phi = Id_{\mathcal{B}_n^{\mathbb{C}}(M)}$, such that $\mathcal{B}_n^{\mathbb{C}}(M), [M, G_n(\mathbb{C}^\infty)]$ are isomorphic. ☐

5.1 Naturality

We have established a bijection between the sets, however we can upgrade this to a natural isomorphism. $\mathcal{B}_n^{\mathbb{C}}(-)$ is a contravariant functor from the category of paracompact spaces to the category of sets, sending each space to the set of isomorphism classes of complex n-bundles over it, with maps given by sending isomorphism classes of a bundle to the isomorphism class of the pullback bundle over the map.
$[-, G_n(\mathbb{C}^\infty)]$ is also a contravariant functor with the same source and target category, when restricted to paracompact spaces.

We have to check the isomorphism between these is natural. So we check the diagram for a map $f : A \to B$

$$\begin{array}{ccc} \mathcal{B}_n^{\mathbb{C}}(B) & \xrightarrow{h_B} & [B, G_n(\mathbb{C}^\infty)] \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \circ f} \\ \mathcal{B}_n^{\mathbb{C}}(A) & \xrightarrow{h_A} & [A, G_n(\mathbb{C}^\infty)] \end{array} \tag{3}$$

commutes. Let $[\omega] \in \mathcal{B}_n^{\mathbb{C}}(B)$ and choose a representative of $[\omega]$ which is a pullback bundle over a map $h : B \to G_n(\mathbb{C}^\infty)$. For $f : A \to B$ we have the following diagram

$$\begin{array}{ccccc} E((h \circ f)^*(\gamma^n)) & \longrightarrow & E(h^*(\gamma^n)) & \longrightarrow & E(\gamma^n) \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{f} & B & \xrightarrow{h} & G_n(\mathbb{C}^\infty). \end{array} \tag{4}$$

The first direction $(\circ f) \circ h_B$ in our naturality diagram, sends the class $[\omega]$ to the class of map $[h]$ and then composing with f, $[h \circ f]$.
The second direction (f^*), sends the class $[h^*(\omega)]$ to the class $[(h \circ f)^*(\gamma^n))]$. Then applying h_A, $[(h \circ f)^*(\gamma^n))]$ get send to the homotopy class of the map of base spaces of a bundle map, which is $[h \circ f]$. This shows the diagram commutes and therefore the functors are naturally isomorphic.

6 Thom Isomorphism

We will here describe a relation between cohomology of the base space and the cohomology of the total space of a vector bundle. For every bundle ξ, there is a deformation retraction from the total space E to the image of the zero section, which is homeomorphic to the base space B. Therefore E and B are homotopy equivalent and $H^i(B) \cong H^i(E)$, but for relative cohomology we get a more interesting isomorphism.

Notation 6.1. *If E is the total space of a vector bundle ξ over B, then we denote by E_0 the complement of the image of the zero section in E, and if V is any vector space, then V_0 denotes the subspace of non-zero elements of V.*

Theorem 6.2 (Thom Isomorphism)**.** *Given a n-bundle ξ, there is an unique $u \in H^n(E, E_0; \mathbb{Z}_2)$, such that for every fiber F, the inclusion $(F, F_0) \hookrightarrow (E, E_0)$ on cohomology*

$$H^n(E, E_0; \mathbb{Z}_2) \to H^n(F, F_0; \mathbb{Z}_2) \cong \mathbb{Z}_2,$$

sends u to the generator $H^n(F, F_0; \mathbb{Z}_2)$.
Furthermore, the map induced by the cup product with u, gives an isomorphism

$$H^j(B; \mathbb{Z}_2) \cong H^j(E; \mathbb{Z}_2) \to H^{j+n}(E, E_0; \mathbb{Z}_2)$$

As in our previous constructions involving cohomology and unoriented bundles, the construction is done with $\mathbb{Z}_2$. For oriented bundles we have a Thom isomorphism with $\mathbb{Z}$ coefficients.

Theorem 6.3 (Oriented Thom Isomorphism)**.** *Let ξ be an oriented n-bundle over B. Then there is a unique $u \in H^n(E, E_0; \mathbb{Z})$, such that for every fiber F, the inclusion $(F, F_0) \hookrightarrow (E, E_0)$ on cohomology*

$$H^n(E, E_0; \mathbb{Z}) \to H^n(F, F_0; \mathbb{Z}) \cong \mathbb{Z},$$

sends u to the generator of $H^n(F, F_0; \mathbb{Z})$ given by the orientation.
Furthermore, the map induced by the cup product with u gives an isomorphism

$$H^j(B; \mathbb{Z}) \cong H^j(E; \mathbb{Z}) \to H^{j+n}(E, E_0; \mathbb{Z}).$$

Notation 6.4. *The element u satisfying the condition of the Thom Isomorphism in both cases, will be called the Thom class. If we want to stress the bundle ξ the Thom class belongs to, we will denote it u_ξ.*

We will prove the theorem in the oriented case and only for compact spaces, but note it holds for all spaces. From now on in this section all cohomology groups will have coefficients in $\mathbb{Z}$. In the case of an oriented trivial bundle we can prove the theorem by considerations of long exact sequences. We denote by $\mathbb{R}_+, \mathbb{R}_-$ the set of positive and negative real numbers respectively.

We have the long exact sequence associated to the triple $(B \times \mathbb{R}, B \times \mathbb{R}_0, B \times \mathbb{R}_-)$. Since $B \times \mathbb{R}$ is homotopy equivalent to $B \times \mathbb{R}_-$, we have $H^i(B \times \mathbb{R}, B \times R_-) = 0$ for all i, and so

$$\partial : H^i(B \times \mathbb{R}_0, B \times \mathbb{R}_-) \to H^{i+1}(B \times \mathbb{R}, B \times \mathbb{R}_0)$$

is an isomorphism for all i, using the long exact sequence. From excision we also have an isomorphism

$$H^i(B \times \mathbb{R}_0, B \times \mathbb{R}_-) \cong H^i(B \times \mathbb{R}_+) \cong H^i(B).$$

In the special case of $B = *$, we can combine our two isomorphisms

$$H^1(\mathbb{R}, \mathbb{R}_0) \cong H^0(\mathbb{R}_0, \mathbb{R}_-) \cong H^0(*).$$

We let $e \in H^1(\mathbb{R}, \mathbb{R}_0)$ be the element, that is the image of $1 \in H^0(\mathbb{R}^+)$ in the above isomorphism. We have the product $e \times e \times ... \times e = e^n \in H^n(\mathbb{R}^n, \mathbb{R}^n_0)$. Here $\mathbb{R}^n_0$ is the vector space $\mathbb{R}^n$ without the zero vector.

Lemma 6.5. *For a space B and a subspace A, the map $\phi : H^j(B, A) \to H^{j+1}(B \times \mathbb{R}, B \times \mathbb{R}_0 \cup A \times \mathbb{R})$, given by $\phi(a) = a \times e$ is an isomorphism for all j.*

Proof. We will first prove the case $A = \emptyset$. For any $a \in H^m(B)$ we get a commutative diagram based on our considerations above

$$\begin{array}{ccccc}
H^0(\mathbb{R}_+) & \xleftarrow{\quad i^* \quad} & H^0(\mathbb{R}_0, \mathbb{R}_-) & \xrightarrow{\quad \partial^* \quad} & H^1(\mathbb{R}, \mathbb{R}_0) \\
\downarrow{\scriptstyle a\times} & & \downarrow{\scriptstyle a\times} & & \downarrow{\scriptstyle a\times} \\
H^m(B \times \mathbb{R}_+) \cong H^m(B) & \xleftarrow{i'^*} & H^m(B \times \mathbb{R}_0, B \times \mathbb{R}_-) & \xrightarrow{\partial'^*} & H^{m+1}(B \times \mathbb{R}, B \times \mathbb{R}_0).
\end{array}$$

The two left horizontal maps are excision isomorphisms, while the two right horizontal maps are ∂^*, ∂'^*, which we showed above are isomorphisms. Going through the diagram with the element $e \in H^1(\mathbb{R}, \mathbb{R}_0)$

$$\begin{array}{ccccc}
1 & \xleftarrow{i^*} & e' & \xrightarrow{\partial^*} & e \\
\downarrow{\scriptstyle a\times} & & \downarrow{\scriptstyle a\times} & & \downarrow{\scriptstyle a\times} \\
a & \xleftarrow{i'^*} & a \times e' & \xrightarrow{\partial'^*} & a \times e,
\end{array}$$

we get that $(\partial'^* \circ (i'^*)^{-1})$ is an isomorphism sending a to $a \times e$. Since this works for all elements of $H^m(B)$ for all $m \in \mathbb{N}$, the map $a \to a \times e$ is equal to the isomorphism $(\partial'^* \circ (i'^*)^{-1})$.

For the general case we have the diagram

$$\begin{array}{ccccc} H^{m-1}(A) & \xrightarrow{\delta} & H^m(B,A) & \longrightarrow & H^m(B) \\ \big\downarrow{\scriptstyle \times e} & & \big\downarrow{\scriptstyle \times e} & & \big\downarrow{\scriptstyle \times e} \\ H^m(A \times \mathbb{R}, A \times \mathbb{R}_0) & \xrightarrow{\delta} & H^m(B \times \mathbb{R}, B \times \mathbb{R}_0 \cup A \times \mathbb{R}) & \longrightarrow & H^{m+1}(B \times \mathbb{R}, B \times \mathbb{R}_0). \end{array}$$

As the previous case shows $\phi : H^j(B) \to H^{j+1}(B \times \mathbb{R}, B \times \mathbb{R}_0)$ and $\phi : H^j(A) \to H^{j+1}(A \times \mathbb{R}, A \times \mathbb{R}_0)$ are isomorphisms for every j, so by the 5-lemma the middle map is also an isomorphism.

□

Using this isomorphism iteratively, we have the composition $\phi : H^j(B) \to H^{j+n}(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0)$ given by $\phi(a) = a \times e^n$ is an isomorphism.

We are now ready to prove Thom's isomorphism theorem. We will first show it for the special case of oriented trivial bundles, and then gradually build up to general oriented bundles.

Proof of Theorem 6.3. In the case of trivial oriented bundle ξ with total space $E \cong B \times \mathbb{R}^n$, we have on cohomology

$$H^n(E, E_0) \cong H^n(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0) \cong H^0(B),$$

by our previous lemma. We have for every fiber the commutative diagram.

$$\begin{array}{ccc} H^0(B) & \xrightarrow{\times e^n} & H^n(E, E_0) \\ \big\downarrow{\scriptstyle i^*} & & \big\downarrow{\scriptstyle i^*} \\ H^0(*) & \xrightarrow{\times e^n} & H^n(F, F_0). \end{array}$$

This shows that finding an element of $H^n(E, E_0)$, that maps to the oriented generator for the inclusion of a fiber, corresponds to finding an element of $H^0(B)$ which maps to 1 for the inclusion of a point. Since only $1 \in H^0(B)$ has this property, we have $u = 1 \times e^n$ is the unique element which satisfies the condition of a Thom class.
To see it induces isomorphism by the cup product, note that all elements of $H^j(B \times \mathbb{R}^n)$, can be written as $y \times 1$ for $y \in H^j(B)$. Again using our previous lemma, the map $y \times 1 \to (y \times 1) \cup (1 \times e^n) = y \times e^n$ is an isomorphism.

Now suppose that we have an oriented bundle ξ over $B = B' \cup B''$, where Thom's isomorphism is true when the bundle is restricted to either B', B'' or $B' \cap B''$. We denote $B' \cap B''$ by $B^\cap$, and the total spaces of $\xi_{|B'}, \xi_{|B''}, \xi_{|B^\cap}$ by $E', E'', E^\cap$ respectively. Associated to the triple of pairs $((E, E_0), (E', E'_0), (E'', E''_0))$ we have the Mayer-Vietoris sequence

$$H^{i-1}(E^\cap, E^\cap_0) \longrightarrow H^i(E, E_0) \xrightarrow{\psi} H^i(E', E'_0) \oplus H^i(E'', E''_0) \xrightarrow{\phi} H^i(E^\cap, E^\cap_0).$$

Since the statement is true for $E', E'', E^\cap$, we have unique elements $u' \in H^n(E', E'_0)$, $u'' \in H^n(E'', E''_0)$ and $u^\cap \in H^n(E^\cap, E^\cap_0)$ which are Thom classes.

For every fiber in $E' \cap E''$, we have the inclusion $(F, F_0) \hookrightarrow (E^\cap, E^\cap_0) \hookrightarrow (E', E'_0)$, so $H^n(E', E'_0) \to H^n(E^\cap, E^\cap_0) \to H^n(F, F_0)$. Since $u^\cap \in H^n(E^\cap, E^\cap_0)$ is the unique element mapping to the oriented generator of $H^n(F, F_0)$ for all fibers, u' must map to $u^\cap$ by the map induced by the inclusion, and similarly u'' must map to $u^\cap$ by the map induced from the inclusion.

Therefore $\phi(u', u'') = u^\cap - u^\cap = 0$. From exactness there exist $u \in H^i(E, E_0)$, with $\psi(u) = (u', u'')$. $\psi(u) = (u', u'')$ is necessary and sufficient for u being a Thom class, since every fiber of E is contained in either E' or E'', and u', u'' are the only elements which will map to the oriented generators.

Using that Thom's isomorphism theorem hold for $\xi_{|B^{\cap}}$, we have $H^{n-1}(E^{\cap}, E_0^{\cap}) \cong H^{-1}(B^{\cap}) = 0$. From the above exact sequence it follows that ψ is injective, so u is the unique element mapping to (u', u'') through ψ.

We are left to prove that the cup product gives an isomorphism. This follows since the cup product induce a map between the two Mayer-Vietoris sequences

$$\begin{tikzcd} H^{i-1-n}(E^{\cap}) \arrow[r] \arrow[d, "\cup u"] & H^{i-n}(E) \arrow[r] \arrow[d, "\cup u"] & H^{i-n}(E') \oplus H^{i-n}(E'') \arrow[d, "\cup u"] \\ H^{i-1}(E^{\cap}, E_0^{\cap}) \arrow[r] & H^{i}(E, E_0) \arrow[r, "\psi"] & H^{i}(E', E_0') \oplus H^{i}(E'', E_0''). \end{tikzcd}$$

Lastly we prove the result for compact spaces. If ξ is an oriented bundle on a compact space B, we can find a finite cover $U_1, ..., U_n$ of B such that $\xi_{|U_i}$ is trivial for all U_i. Note that the bundle must also be trivial on the intersections of U_i, since they are restrictions of trivial bundles. We prove inductively that Thom's isomorphism is true for $\xi_{|U_1 \cup ... \cup U_m}$. Since $\xi_{|U_1}$ is trivial, Thom's isomorphism is true in this case.
If we assume it is true for $\xi_{|U_1 \cup ... \cup U_{m-1}}$, we have $U_1 \cup ... \cup U_m = (U_1 \cup ... \cup U_{m-1}) \cup U_m$, and since we know it is true for both sets in the union, and also true for the intersection since it is trivial, the result must be true for $\xi_{|U_1 \cup ... \cup U_m}$, so Thom isomorphism theorem is also true for ξ. ☐

We have now proved the statement for all compact spaces, the proof for the general case can be found here[5, p. 110].

7 Stiefel-Whitney Classes

We can now define characteristic classes of vector bundles.

Definition 7.1 (Characteristic Classes). *A characteristic class a of unoriented n-bundles in degree q with coefficient group π, is a natural transformation $\mathcal{B}_n(-) \to H^q(-, \pi)$. Explicitly it sends a vector bundle ξ over a space B, to an element $a(\xi) \in H^q(B, \pi)$. Oriented characteristic classes are defined the same way with source functor $\mathcal{B}_n^+$, and for complex characteristic classes with source functor $\mathcal{B}_n^{\mathbb{C}}$.*

From our natural isomorphism $\mathcal{B}_n(-) \overset{\text{iso}}{\rightsquigarrow} [-, G_n(\mathbb{R}^\infty)]$, $\mathcal{B}_n(-)$ is a representable functor, so from the Yoneda lemma we get a bijection between natural transformations in degree q with coefficients π and $H^q(G_n(\mathbb{R}^\infty), \pi)$. It follows that we can classify unoriented characteristic classes by studying $H^q(G_n(\mathbb{R}^\infty), \pi)$, and similarly for oriented and complex characteristic classes with $\tilde{G}_n(\mathbb{R}^\infty), G_n(\mathbb{C}^\infty)$ respectively.

7.1 Stiefel-Whitney classes

We are now ready to define our first characteristic class defined for all dimensions and degrees.

Definition 7.2 (Stiefel-Whitney classes). *The Stiefel-Whitney w_i class of an unoriented bundle is of degree i with coefficients $\mathbb{Z}_2$. It upholds the axioms*
1) $w_0(\xi) = 1$ and $w_i(\xi) = 0$ for $i > n$ if ξ is a n-plane bundle.
2) $w_i(\xi \oplus \eta) = \sum_j^n w_j(\xi) \oplus w_{i-j}(\eta)$
3) $w_1(\gamma_1)$ is non-zero, where γ_1 is the tautological line bundle over $\mathbb{R}P^\infty$

The Stiefel-Whitney classes are defined uniquely from these axioms.[5, p. 86]
Given a graded ring A, we let A^Π be the set of sequences $(a_n)_{n \in \mathbb{N}_0}$ where a_n is of degree n and $a_0 = 1$. It is a ring with degree-wise addition, and multiplication given by

$$(1 + a_1 + a_2 + ...)(1 + b_1 + b_2 + ...) = 1 + (a_1 + b_1) + (a_1 b_1 + a_2 + b_2) + ...$$

Applying this to $(H^*(B, \mathbb{Z}_2))^\Pi$, we can define the total Stiefel-Whitney class $w = (w_i)_{i \in \mathbb{N}_0} \in (H^*(B, \mathbb{Z}_2))^\Pi$. This gives us an easier product rule $w(\xi \oplus \eta) = w(\xi)w(\eta)$. We can now calculate the Stiefel Whitney classes of some examples.

The trivial bundle has total class $w(\epsilon^n) = 1$, since it is the pullback bundle of a n-bundle over a point, which has no cohomology in dimensions higher than 0.
If we take the Whitney sum of the tangent bundle of S^n with the outward pointing vector field in the embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$, we get the trivial $(n+1)$-bundle. It follows $w(\tau^n) = w(\tau^n \oplus \epsilon) = w(\epsilon^{n+1}) = 1$ on the n-sphere.
This example shows the Stiefel-Whitney classes does not give us all information about vector bundles, as not all tangent bundles of spheres are trivial. In the limited case of 1-bundles they do however classify all bundles.

Theorem 7.3. *w_1 gives a natural isomorphism between $\mathcal{B}_1(-)$ and $H^1(-, \mathbb{Z}_2)$.*

Proof. We have the surprising fact, that $\mathbb{R}\mathrm{P}^\infty$ is not only the classifying space of 1-bundles but also the Eilenberg-MacLane space $K(\mathbb{Z}_2, 1)$, giving us that $H^1(-, \mathbb{Z}_2) \overset{\text{iso}}{\rightsquigarrow} [-, \mathbb{R}\mathrm{P}^\infty]$. It follows we have

$$[-, \mathbb{R}\mathrm{P}^\infty] \overset{\text{iso}}{\rightsquigarrow} \mathcal{B}_1(-) \overset{w_1}{\rightsquigarrow} H^1(-, \mathbb{Z}_2) \overset{\text{iso}}{\rightsquigarrow} [-, \mathbb{R}\mathrm{P}^\infty]$$

From the Yoneda lemma, we only have to show that $\mathrm{Id}_{\mathbb{R}\mathrm{P}^\infty}$ gets sent to itself, to see this natural transformation is the identity, which will imply w_1 is a natural isomorphism.

First $\mathrm{Id}_{\mathbb{R}\mathrm{P}^\infty}$ corresponds to γ^1, since $(\mathrm{Id}_{\mathbb{R}\mathrm{P}^\infty})^*(\gamma^1) = \gamma^1$. From our assumption that $w_1(\gamma^1)$ is non-zero, it must be the unit of $H^1(\mathbb{R}\mathrm{P}^\infty)$ since it is isomorphic to $\mathbb{Z}_2$. Composing with $H^1(-, \mathbb{Z}_2) \overset{\text{iso}}{\rightsquigarrow} [-, \mathbb{R}\mathrm{P}^\infty]$ it sends the identity to the identity map $\mathrm{Id}_{\mathbb{R}\mathrm{P}^\infty}$, showing the whole natural transformation to be the identity. This shows that w_1 is a natural isomorphism. □

7.2 Calculation of $w(\tau^n)$ for $\mathbb{R}\mathrm{P}^n$

The calculation of the characteristic classes of the tangent bundles of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ will be an important piece in uncovering the question of cobordisms.
As in the calculation of Stiefel-Whitney classes of the tangent bundle of S^n, we can take Whitney sums to get another bundle whose characteristic class we already now. Note that given two bundles ξ, η over the same base space, we can get a new bundle $\hom(\xi, \eta)$, which on fibers $b \in B$ is the vector space $\hom(\xi_b, \eta_b)$. The proof that this defines a bundle can be found here[5, p. 31]. Note that as the fibers of γ_n^1 is a n-plane inside $\mathbb{R}^{n+1}$, we have γ_n^1 is a subbundle of ϵ^{n+1}, and so we have the orthogonal bundle of this inclusion $\gamma_n^{1\perp}$.

Lemma 7.4. *The tangent bundle τ^n on $\mathbb{R}\mathrm{P}^n$ is isomorphic to* $\hom(\gamma_n^1, \gamma_n^{1\perp})$

Proof. [5, p. 43]. □

The following lemma extends two properties of hom from vector spaces to vector bundles.

Lemma 7.5. *For every bundle ξ over a paracompact space* $\hom(\xi, \epsilon) \cong \xi$ *and* $\hom(\xi, \xi) \cong \epsilon$ *if ξ is a 1-bundle.*

Proof. We have $\hom(\xi_b, \epsilon_b) \cong \xi_b$ is isomorphic as vector spaces but the isomorphism is not canonical. However since the vector bundle can be given an euclidean structure, we have the musical[4] isomorphism $\xi_b \to \hom(\xi_b, \epsilon_b)$ by $v \to \langle -, v \rangle$. Since an inner product is positive definite and linear, it gives a canonical isomorphism, which then can be extended to an isomorphism $\xi \cong \hom(\xi, \epsilon)$. Note that the isomorphism depends on the choice of euclidean structure.

Assuming ξ is a 1-bundle, $\hom(\xi, \xi) \cong \epsilon$ since it is a 1-bundle with non-zero section given by the identity map on fibers.

□

Theorem 7.6. *The tangent bundle τ^n on $\mathbb{R}\mathrm{P}^\infty$ has the relation $\tau^n \oplus \epsilon \cong (\gamma_n^1)^{n+1}$.*

Proof. From our lemmas we get

$$\tau^n \oplus \epsilon \cong \hom(\gamma_n^1, \gamma_n^{1\perp}) \oplus \hom(\gamma_n^1, \gamma_n^1) \cong \hom(\gamma_n^1, \gamma_n^1 \oplus \gamma_n^{1\perp}) \cong \hom(\gamma_n^1, \epsilon^{n+1})$$

[4] An explanation of the name can be found here[2, p. 342].

This can again be split into $n+1$ bundles

$$\cong \hom(\gamma_n^1, \epsilon)^{n+1} \cong (\gamma_n^1)^{n+1}$$

□

We can now calculate $w(\tau^n)$ by

$$w(\tau^n) = w(\gamma_n^1)^{n+1} = (1+a)^{n+1},$$

where a is the nonzero element in $H^1(\mathbb{R}\mathrm{P}^n, \mathbb{Z}_2)$.
From this we can conclude some real projective spaces can not be parallelizable.

Corollary 7.7. *The only real projective space which can be parallelizable is $\mathbb{R}\mathrm{P}^n$, where $n = 2^k - 1$ for some integer k.*

Proof. In the case $n = 2^k - 1$ we have $w(\tau) = (1+a)^{n+1} = (1+a)^{2^k}$. Using that $(1+a)^2 = 1 + a^2$ in modulo 2 inductively, we get $w(\tau) = (1+a)^{n+1} = (1+a)^{2^k} = 1 + a^{2^k} = 1$ since $a^{n+1} = 0$.

If $n = 2^k m - 1$ with m odd and at greater than 2, we have $w(\tau) = (1+a)^{2^k m} = (1 + a^{2^k})^m = 1 + ma^{2^k} ... + ma^{2^k(m-1)}$. Since m, the first and last coefficient is non-zero.

Therefore the Stiefel-Whitney class only agree with the trivial bundle when $n = 2^k - 1$. □

7.3 Stiefel-Whitney Numbers

Since the Stiefel Whitney classes are elements of the cohomology, we can try and evaluate homology on the classes. Specifically since any manifold is $\mathbb{Z}_2$-orientable, we have for a compact manifold a fundamental class $[M] \in H_n(M; \mathbb{Z}_2)$, which we can evaluate on[4, p. 155].

Definition 7.8 (Stiefel-Whitney Number)**.** *Given a compact n-manifold and an unordered partition $J = i_1 i_2 ... i_m$ of n into positive integers, the J-th Stiefel-Whitney number is*

$$\langle w_{i_1}(\tau^n) w_{i_2}(\tau^n) \cdots w_{i_m}(\tau^n), [M] \rangle \in \mathbb{Z}_2,$$

where $\langle , \rangle$ is the evaluation of cohomology on homology.

The Stiefel-Whitney numbers are invariants of manifolds, and they will give us our first connection between the theory of vector bundles and cobordisms, seen by the next theorem.

Theorem 7.9. *If a compact manifold is nul-cobordant then all of its Stiefel-Whitney numbers are 0.*

Proof. Assume that a compact manifold M is a boundary of a compact manifold W with boundary. Then there exist a fundamental class $[W] \in H_{n+1}(W, M; \mathbb{Z}_2)$ such that $\partial[W] = [M]$[4, p. 168].

We would like to induce the Stiefel-Whitney classes of the tangent bundle on M by the map $i : M \to W$ but the inclusion of tangent bundles does not give a bundle map. However since every tangent bundle has an outward pointing vector field on the boundary, there exist a trivial subbundle of dimension 1 which is in direct sum with τ^n. It follows that we have a bundle map

$$\begin{tikzcd} E(\tau_M^n \oplus \epsilon^1) \arrow[r] \arrow[d] & E(\tau_W^n) \arrow[d] \\ M \arrow[r, "i"] & W, \end{tikzcd}$$

and so $i^*(w_j(\tau_W^{n+1})) = w_j(\tau_M^n \oplus \epsilon^1) = w_j(\tau_M^n)$.
From this we have the calculation of the J-th Stiefel-Whitney number

$$\begin{aligned} \langle w_{i_1}(\tau_M^n) w_{i_2}(\tau_M^n) \cdots w_{i_m}(\tau_M^n), \partial_*([W]) \rangle &= \langle i^*(w_{i_1}(\tau_W^{n+1}) w_{i_2}(\tau_W^{n+1}) \cdots w_{i_m}(\tau_W^{n+1})), \partial_*([W]) \rangle \\ &= \langle w_{i_1}(\tau_W^{n+1}) w_{i_2}(\tau_W^{n+1}) \cdots w_{i_m}(\tau_W^{n+1}), (i^* \circ \partial)([W]) \rangle = 0. \end{aligned}$$

Since $(i \circ \partial)_* = 0$ from the long exact sequence

$$H^{n+1}(W, M; \mathbb{Z}_2) \xrightarrow{\partial} H^n(M; \mathbb{Z}_2) \xrightarrow{i_*} H^n(W; \mathbb{Z}_2).$$

□

Actually the reverse implication is also true; a manifold is nul-cobordant if all of the Stiefel-Whitney numbers are 0[5][4, p. 228].

7.4 Euler Class

For oriented bundles we get a characteristic class from the oriented Thom isomorphism.

Definition 7.10 (Euler Class). *Let ξ be an oriented bundle with fundamental cohomology class $u \in H^n(E, E_0)$. The image of u along the map $H^n(E, E_0) \to H^n(E) \cong H^n(B)$ is the Euler class $u_{|E}$ denoted $e(\xi)$.*

This construction is natural, since if we have a oriented bundle map (g, f) from ξ to η then we have a diagram of oriented maps for every $x \in B(\xi)$

$$\begin{array}{ccc} (F(\xi), F(\xi)_0) & \xrightarrow{(g_{|F}, g_{|F_0})} & (F(\eta), F(\eta)_0) \\ \Big\downarrow i_x & & \Big\downarrow i_{f(x)} \\ (E(\xi), E(\xi)_0) & \xrightarrow{(g, g_0)} & (E(\eta), E(\eta)_0). \end{array}$$

From this we get $i_x^*(g^*(u_\eta))$ is the oriented generator of $H^n(F(\xi), F(\xi)_0)$ for every $x \in B(\xi)$, so from uniqueness of the Thom class we have $g^*(u_\eta) = u_\xi$.

As with Stiefel-Whitney classes, Euler classes are multiplicative in the sense $e(\xi \oplus \eta) = e(\xi)e(\eta)$, and if ξ has a non-zero section then $e(\xi) = 0$[5, p. 98].
As with Stiefel-Whitney numbers, for an oriented n-manifold M we have the number $\langle e(\tau_M), \mu \rangle$, where $\mu \in H^n(M, \mathbb{Z})$ is the fundamental class. Even though $\langle e(\tau_M), \mu \rangle$ is constructed using the smooth structure of M, it can be shown that it coincides with the Euler characteristic of M[5, p. 130], which explains the name of the characteristic class.

8 Chern Classes

The most important characteristic classes of complex vector bundles are the Chern classes c_i. To construct Chern classes we first construct a way to get a $(n-1)$-bundle from a n-bundle.

Definition 8.1. *Let ω be a complex n-bundle. For $v \in E_0$, let $F_{\pi(v)}$ be the fiber of $\pi(v)$ in ω. Then ω_0 is a $(n-1)$-bundle on $E_0(\omega)$, with fiber of $v \in E_0$ being $F_{\pi(v)}/(\mathbb{C}v)$, where $\mathbb{C}v$ is the linear subspace spanned by v in E.*

We have for an oriented $2n$-bundle the Gysin sequence, obtained by combining the long exact sequence of (E, E_0) with the Thom isomorphism theorem.

$$H^{i-2n}(B) \xrightarrow{\cup e(\omega)} H^i(B) \xrightarrow{\pi_0^*} H^i(E_0) \longrightarrow H^{i-2n+1}(B)$$

For $i < 2n - 1$ the outer groups are 0, so we get isomorphisms. We can now define Chern classes.

Definition 8.2 (Chern Classes). *We define Chern classes inductively on the dimension on the bundle. For a complex n-bundle ω, $c_i = 0$ for $i > n$ and $c_n = e(\omega) \in H^{2n}(B)$. Since complex bundles are oriented this makes sense. For $i < n$, $c_i(\omega) = \pi_0^{-1}(c_i(\omega_0)) \in H^{2i}(B)$ from the inductive assumption.*

Lemma 8.3. *The construction of Chern classes is natural, explicitly if there is a bundle map $\omega \to \omega'$ over a map $g : B \to B'$, then $g^*(c_i(\omega')) = c_i(\omega)$ for any i.*

Proof. We will proceed by induction.
For 1-bundles it is true since the only Chern class c_1 is the Euler class, which we know is natural.
Assuming it is true for dimension lower than n, let ω, ω' be n-bundles with a bundle map (f, g).
For $v \in E_0(\omega)$ with $\pi(v) = b$ the map

$$f_v : F_b/(\mathbb{C}v) \to F_{g(b)}/(\mathbb{C}f(v))$$

[5]The same idea as here, will be used later in our discussion of oriented cobordism.

given by the restriction of f to the fiber b along with taking quotients, is linear and bijective. Let $f' : E(\omega_0) \to E(\omega_0')$ be the map induced by f_v on the fiber $v \in E(\omega_0)$. Then $(f', f_{|E_0(\omega)})$ is a bundle map of $(n-1)$-bundles. From the induction assumption we have $c_i(\omega_0) = f'^*(\omega_0')$. From the diagram

$$\begin{array}{ccc} H^i(E_0(\omega)) & \xleftarrow[f'^*]{} & H^i(E_0(\omega')) \\ {\scriptstyle \pi_0^*}\uparrow & & {\scriptstyle \pi_0^*}\uparrow \\ H^i(B(\omega)) & \xleftarrow[g^*]{} & H^i(B(\omega')), \end{array}$$

we get that $c_i(\omega_0) = \pi_0^{*-1}(f'^*(c_i(\omega_0'))) = g^*(c_i(\omega')$, showing naturality. $\square$

As with Stiefel-Whitney classes, the Chern classes of the trivial complex bundles are 0, since we have a bundle map over the map to a point.
We have the total Chern class

$$1 + c_1(\omega) + c_2(\omega) + ... + c_n(\omega),$$

which also has a product rule $c(\omega \oplus \omega') = c(\omega)c(\omega')$[5, p. 164]. For a complex bundle ω, the Chern classes of $\overline{\omega}$ are given by $c_i(\overline{\omega}) = (-1)^i c_k(\omega)$ [5, p. 168]. Under the coefficient change in cohomology $\mathbb{Z} \to \mathbb{Z}_2$, the total Chern class get mapped to the total Stiefel-Whitney class, in particular c_i get send to w_{2i} [5, p. 171].

8.1 Calculations for the $\mathbb{C}\mathrm{P}^n$

Even though the definition of Chern classes is hard to wrap your head around, we can still make calculations by finding the Euler class, which we have different tools to deal with, and by using Chern classes are multiplicative. The tautological bundle and the tangent bundle on $\mathbb{C}\mathrm{P}^n$ illustrates this approach. First we will look at the tautological bundle γ_n^1 on $\mathbb{C}\mathrm{P}^n$.

Since the bundle is 1-dimensional, the only possible non-zero Chern class is $c_1(\gamma_n^1) = e(\gamma_n^1)$. Again we have the Gysin sequence

$$H^{i-1}(E_0) \longrightarrow H^{i-2}(\mathbb{C}\mathrm{P}^n) \xrightarrow{\cup e(\gamma_n^1)} H^i(\mathbb{C}\mathrm{P}^n) \xrightarrow{\pi_0^*} H^i(E_0).$$

Note that E_0 consists of (l, v), where l is a 1-dimensional subspace in $\mathbb{C}^{n+1}$ and v is a non-zero vector with $v \in l$. As every non-zero vector in $\mathbb{C}^{n+1}$ only lies in one complex line, we have $E_0 \cong \mathbb{C}^{n+1} - 0 \simeq S^{2n+1}$. Using this, our sequence becomes

$$0 \longrightarrow H^i(\mathbb{C}\mathrm{P}^n) \xrightarrow{\cup e(\gamma_n^1)} H^{i+2}(\mathbb{C}\mathrm{P}^n) \xrightarrow{\pi_0^*} 0,$$

when $2 \leq i \leq 2n$. It follows that $\mathbb{Z} \cong H^0(\mathbb{C}\mathrm{P}^n) \cong H^{2i}(\mathbb{C}\mathrm{P}^n)$ for $0 \leq i \leq n$ and that $H^{2i}(\mathbb{C}\mathrm{P}^n)$ has generator $e(\gamma_n^1)^i \in H^{2i}(\mathbb{C}\mathrm{P}^n)$. For odd degrees we have $0 \cong H^{-1}(\mathbb{C}\mathrm{P}^n) \cong H^{2i-1}(\mathbb{C}\mathrm{P}^n)$ for $0 \leq i \leq n$, and since $\mathbb{C}\mathrm{P}^n$ is a $2n$-dimensional manifold, cohomology vanishes in every other degree. By the inclusions $\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^\infty$ we get isomorphism on cohomoloy degree less than $2n$, so letting n to to ∞ we get that $H^*(\mathbb{C}\mathrm{P}^\infty)$ is the free polynomial ring generated by the Euler class of the tautological bundle.

For the tangent bundle, by a similar argument to Theorem 7.6, we get the relation $\tau^{2n} \oplus \epsilon \cong (\overline{\gamma}_n^1)^{n+1}$ and by the product rule we then have

$$c(\tau^{2n}) = c(\tau^{2n} \oplus \epsilon) = c(\overline{\gamma}_n^1)^{n+1} = (1 - e(\gamma_n^1))^{n+1} = (1 + a)^{n+1},$$

where $a = -e(\gamma_n^1)$, so $c_i(\tau^{2n}) = \binom{n+1}{i} e(\gamma_n^1)^i$.
Thus $e(\tau^{2n}) = (n+1)a^n$, and using that $\langle e(\tau^{2n}), \mu \rangle = \chi(\mathbb{C}\mathrm{P}^{2n}) = n+1$, we get that $\langle a^n, \mu \rangle = 1$, so a^n is the oriented generator of $\mathbb{C}\mathrm{P}^{2n}$.

Lastly we will note that the calculation of cohomology of complex projective space generalizes to Grassmannians in higher dimension.

Theorem 8.4. *For $G_n(\mathbb{C}^\infty)$ we have $H^*(G_n(\mathbb{C}^\infty))$ is the free polynomial ring generated by $c_1(\gamma^n), c_2(\gamma^n), ..., c_n(\gamma^n)$.*

The proof can be found here[5, p. 161].

9 Pontryagin Classes

While the Chern classes is only defined for complex bundles, we can use them to define the Pontryagin classes for all real bundles.

Recall that by complexification we can turn every real bundle into a complex bundle. A natural construction would then be to take Chern classes of the complexification of a bundle ξ. However from Lemma 4.17 we know that

$$c_i(\xi \otimes \mathbb{C}) = c_i(\overline{\xi \otimes \mathbb{C}}) = (-1)^i c_i(\xi \otimes \mathbb{C})$$

So when i is odd, $c_i(\xi \otimes \mathbb{C})$ is an element of order 2. We will instead focus on the even Chern classes, which can be of infinite order.

Definition 9.1 (Pontryagin Classes)**.** *For a n-bundle ξ, the i'th Pontryagin classes is $(-1)^i c_{2i}(\xi \otimes \mathbb{C}) \in H^i(B, \mathbb{Z})$, with a sign introduced to make later statements easier to write.*

As with previous cases we have the total Pontryagin class for an n-bundle

$$p(\xi) = p_0(\xi) + p_1(\xi) + ... + p_{\lfloor n/2 \rfloor}(\xi)$$

As the Pontryagin Classes are constructed from Chern classes, they have a lot of the same properties, when the information lost by the odd Chern classes are disregarded.

As the complexification of a trivial -bundle ϵ^n is the trivial complex n-bundle, we have $p_0(\epsilon^n) = 1$ and $p_i(\epsilon^n) = 0$ in every other degree. We also have $(\xi \oplus \eta) \otimes \mathbb{C} \cong (\xi \otimes \mathbb{C}) \oplus (\eta \otimes \mathbb{C})$, so we get

$$p_i(\xi \oplus \eta) = (-1)^i c_{2i}((\xi \otimes \mathbb{C}) \oplus (\eta \otimes \mathbb{C})) = (-1)^i \sum_{j=0}^{2i} c_{2i-j}(\xi \otimes \mathbb{C}) c_j(\eta \otimes \mathbb{C})$$

We let a be the part of the sum $\sum_{j=0}^{i-1} c_{2i-(2j+1)}(\xi \otimes \mathbb{C}) c_{2j+1}(\eta \otimes \mathbb{C})$. since each $c_{2i-(2j+1)}(\xi \otimes \mathbb{C}) c_{2j+1}(\eta \otimes \mathbb{C})$ is a product of two odd Chern classes, the sum has order two. Using $(-1)^i = (-1)^j(-1)^{i-j}$ we have

$$a + \sum_{j=0}^{i} (-1)^{i-j} c_{2(i-j)}(\xi \otimes \mathbb{C})(-1)^j c_{2j}(\eta \otimes \mathbb{C}) = a + \sum_{j=0}^{i} p_{i-j}(\xi) p_j(\eta)$$

Showing we have a product rule for Pontryagin classes up to an element of order 2. Even though this is weaker than our earlier product rules, we can take the coefficients in $\mathbb{Q}$ to remove the torsion from the product rule.

When we take the Pontryagin class of a complex n-bundle ω, we can use Lemma 4.18 together with the calculation of Chern classes of conjugate bundles.

$$(-1)^i p_i(\omega) = c_i(\omega \otimes \mathbb{C}) = c_i(\omega \oplus \overline{\omega}) = \sum_{j=0}^{i} c_{i-j}(\omega)(-1)^j c_j(\omega)$$

Taking the sum of these we get

$$\sum_{i=0}^{n} (-1)^i p_i(\omega) = \sum_{i=0}^{n} \sum_{j=0}^{i} c_{i-j}(\omega)(-1)^j c_j(\omega) = \left(\sum_{i=0}^{n} c_i(\omega) \right) \left(\sum_{i=0}^{n} (-1)^i c_i(\omega) \right)$$

9.1 Calculation of Pontryagin Classes

Given these tools we can calculate the Pontryagin classes of the tangent bundle on $\mathbb{CP}^n$, since we know their Chern classes.

$$\sum_{i=0}^{n} (-1)^i p_i(\tau^n) = \left(\sum_{i=0}^{n} c_i(\tau^n) \right) \left(\sum_{i=0}^{n} (-1)^i c_i(\tau^n) \right) = (1+a)^{n+1}(1-a)^{n+1} = (1-a^2)^{n+1} = \sum_{i=0}^{n} (-1)^i \binom{n+1}{i} a^{2i}$$

Removing the $(-1)^i$ from both sides we get

$$\sum_{i=0}^{n} p_i(\tau^n) = \sum_{i=0}^{n} \binom{n+1}{i} a^{2i}$$

We can describe the cohomology of $\tilde{G}_n(\mathbb{R}^\infty)$ with rational coefficients by the Pontryagin classes of the tautological bundle.

Theorem 9.2. *If n is odd $H^*(\tilde{G}_n(\mathbb{R}^\infty);\mathbb{Q})$ is the free polynomial ring generated by the Pontryagin classes*

$$p_1(\tilde{\gamma}^n), p_2(\tilde{\gamma}^n), ..., p_{\lfloor \frac{n}{2} \rfloor}(\tilde{\gamma}^n)$$

If n is even, $H^*(\tilde{G}_n(\mathbb{R}^\infty);\mathbb{Q})$ is the free polynomial ring generated by the Pontryagin classes

$$p_1(\tilde{\gamma}^n), p_2(\tilde{\gamma}^n), ..., p_{\frac{n}{2}-1}(\tilde{\gamma}^n), e(\tilde{\gamma}^n)$$

with $e(\tilde{\gamma}^n)^2 = p_{\frac{n}{2}}(\tilde{\gamma}^n)$

The proof can be found here[5, p. 179].

10 Pontryagin Numbers

For a closed oriented n-manifold M, we have the fundamental class $\mu_M \in H_n(M,\mathbb{Z})$. As with Stiefel-Whitney numbers we can pair the Pontryagin classes of the tangent bundle with μ_M, however as the Pontryagin classes are elements of $H^{4m}(M)$, we will restrict to manifolds of dimension divisible of 4.

Definition 10.1 (Pontryagin Number)**.** *Given a compact $4n$-manifold and a unordered partition $I = i_1 i_2 ... i_m$ of n into positive integers, the I-th Pontryagin number is*

$$\langle p_{i_1}(\tau^{4n}) p_{i_2}(\tau^{4n}) \cdots p_{i_m}(\tau^{4n}), [M] \rangle \in \mathbb{Z},$$

where $\langle , \rangle$ is the evaluation on cohomology on homology.

The Pontryagin number are additive with respect to disjoint union, since Pontryagin classes and fundamental classes of a disjoint union are equal to the sum of their parts, and for any pair of manifolds M, M'

$$\langle p_{i_1}(\tau_M^{4n}) p_{i_2}(\tau_M^{4n}) \cdots p_{i_m}(\tau_M^{4n}), [M'] \rangle = 0$$

Inside $M \amalg M'$.
These numbers are powerful invariants of oriented closed manifolds, and from the following theorem they respect cobordisms.

Theorem 10.2. *If a closed oriented manifold is oriented nul-cobordant then all of its Pontryagin numbers are 0.*

Proof. The proof is similar to the one given for Stiefel-Whitney numbers. Assume that an oriented closed manifold M is the boundary of an oriented compact manifold W with boundary. Then there exist a fundamental class $[W] \in H_{n+1}(W, M, \mathbb{Z})$ such that $\partial[W] = [M]$[4, p. 170].

The Pontryagin classes of the tangent bundle on M is induced by the Pontryagin classes of W since

$$\begin{array}{ccc} E(\epsilon^1 \oplus \tau_M^n) & \longrightarrow & E(\tau_W^n) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\quad i \quad} & W, \end{array}$$

and so $i^*(p_j(\tau_W^{n+1})) = p_j(\tau_M^n \oplus \epsilon^1) = p_j(\tau_M^n)$.
From this we have the calculation of the I-th Pontryagin number

$$\langle p_{i_1}(\tau_M^n) p_{i_2}(\tau_M^n) \cdots p_{i_m}(\tau_M^n), \partial_*([W]) \rangle = \langle i^*(p_{i_1}(\tau_W^{n+1}) p_{i_2}(\tau_W^{n+1}) \cdots p_{i_m}(\tau_W^{n+1})), \partial_*([W]) \rangle$$

$$= \langle p_{i_1}(\tau_W^{n+1}) p_{i_2}(\tau_W^{n+1}) \cdots p_{i_m}(\tau_W^{n+1}), (i^* \circ \partial)([W]) \rangle = 0.$$

Since $(i \circ \partial)_* = 0$ from the long exact sequence

$$H^{n+1}(W, M) \xrightarrow{\partial} H^n(M) \xrightarrow{i_*} H^n(W).$$

$\square$

From this we get that Pontryagin numbers are group homomorphisms from Ω_n^+ to $\mathbb{Z}$. By calculating the Pontryagin numbers of even complex projective spaces we can show they are nulcobordant.

For a partition $I = i_1 i_2 ... i_m$ of n, the I-th Pontryagin number of $\mathbb{C}\mathrm{P}^{2n}$ is given by

$$\langle p_{i_1}(\tau^{4n}) p_{i_2}(\tau^{4n}) \cdots p_{i_m}(\tau^{4n}), [\mathbb{C}\mathrm{P}^{2n}] \rangle = \binom{2n+1}{i_1} \binom{2n+1}{i_2} \cdots \binom{2n+1}{i_m} \langle a^{n+1}, [\mathbb{C}\mathrm{P}^{2n}] \rangle.$$

Since $\langle a^{n+1}, [\mathbb{C}\mathrm{P}^{2n}] \rangle = 1$, the I-th Pontryagin number of $\mathbb{C}\mathrm{P}^{2n}$ is equal to $\binom{2n+1}{i_1}\binom{2n+1}{i_2} \cdots \binom{2n+1}{i_m}$. Since the Pontryagin numbers are additive, we can conclude that $[\mathbb{C}\mathrm{P}^{2n}]$ is infinitely cyclic in Ω_{4n}^+. Since $\mathbb{C}\mathrm{P}^{2n+1}$ is not of dimension divisible by 4, we can not extend the conclusion to these spaces, and these spaces are actually nul-cobordant.

By the following theorem of Thom we can extend this result to show that the products of even complex projective spaces generate a rank $\kappa(n)$ subgroup of Ω_{4n}^+, where $\kappa(n)$ is the number of partitions of n. The $\kappa(n)$ products of even complex projective spaces of total dimension $4n$, generate the subgroup $A \hookrightarrow \Omega_{4n}^+$.

Theorem 10.3. *Let $\bigtimes_{I, |I|=n} \mathbb{Q}$ be the product of $\mathbb{Q}$ over the partitions of n. The map*

$$A \otimes \mathbb{Q} \to \bigtimes_{I, |I|=n} \mathbb{Q}$$

given in the I'th coordinate by $[M] \otimes q \to q \cdot p_I(M)$ is an isomorphism.

From this we can conclude that Ω_{4n}^+ has at least rank $\kappa(n)$.

11 Pontryagin-Thom Construction

We are now ready to define the Thom space of a vector bundle.

Definition 11.1 (Thom Space)**.** *Let ξ be a smooth euclidean bundle. Then $D(\xi) \subset E(\xi)$ is the subspace consisting of the vectors v with $|v| \leq 1$, and furthermore $S(\xi) \subset D(\xi)$ is the subspace of vectors v with $|v| = 1$. The Thom space $T(\xi)$ is a pointed space $D(\xi)/S(\xi)$ with basepoint $t_0 = [S(\xi)]$.*

Notation 11.2. *For an euclidean bundle ξ, we let $D_0(\xi) = E_0(\xi) \cap D(\xi)$, $S_0(\xi) = E_0(\xi) \cap S(\xi)$ and $T_0(\xi) = D_0(\xi)/S_0(\xi)$. We will just write T, T_0 and so on for these constructions, when the bundle is clear from the context.*

We will note some useful facts about Thom spaces for a bundle ξ. Firstly the space $T(\xi) \setminus \{t_0\}$ is diffeomorphic to $E(\xi)$, by scaling vectors v by $\frac{|v|}{1-|v|}$. If ξ is over a compact space, $T(\xi)$ is the one-point compactification of $E(\xi)$. The zero-section $i : B \to E(\xi)$ induces an embedding $i' : B \to T(\xi)$ also referred to as the zero-section of $T(\xi)$.

We can also calculate the cohomology of $T(\xi)$ relative to the basepoint by using the Thom isomorphism.

Lemma 11.3. *If ξ is an oriented n-bundle over a space B, then $H^{n+i}(T(\xi), t_0) \cong H^i(B)$*

Proof. We have a deformation retract of D_0 onto S_0 by

$$h(v,t) = \left(1 + t\left(\frac{1}{|v|} - 1\right)\right) \cdot v.$$

Sending the homotopy to the quotient, we get a deformation retract of T_0 onto t_0, which gives

$$H^{n+i}(T, t_0) \cong H^{n+i}(T, T_0)$$

Using excision on the triad $(T, T_0, T - t_0)$, along with the Thom isomorphism we get

$$H^{n+i}(T, T_0) \cong H^{n+i}(T - t_0, T_0 - t_0) \cong H^{n+i}(E, E_0) \cong H^i(B).$$

□

11.1 Manifolds from Regular Maps

Given manifolds M, N and a smooth map $f : M \to N$ between them, we are not guaranteed that the preimage $f^{-1}(y) \in M$ of a point $y \in N$, is a smooth submanifold of M. For example the map $g : \mathbb{R}^2 \to \mathbb{R}$ given by $g(a, b) = a^2 - b^2$, has the preimage $g^{-1}(0)$ homeomorphic to $\mathbb{R} \vee \mathbb{R}$ which is not a manifold. We have however a condition, that ensures the preimage is a manifold.

Definition 11.4. *Let M, N be manifolds with a smooth map $f : M \to N$. A point $y \in N$ is called a regular value, if for every point $x \in f^{-1}(y)$ the induced map on tangent spaces*

$$TM_x \xrightarrow{Tf_x} TN_y$$

is surjective. A point in N is a critical value, if it is not regular. If X is a subset of M such that the condition is satisfied for $x \in f^{-1}(y) \cap X$, then y is regular throughout X.

If we have a smooth submanifold $Y \subset N$ then f is transverse to Y if the map

$$TM_x \xrightarrow{Tf_x} TN_y \longrightarrow TN_y/TY_y$$

is surjective for every $x \in f^{-1}(Y)$. We also call f transverse to Y throughout $X \in M$ if the condition holds for $f^{-1}(Y) \cap X$.

If we regard a point $y \in N$ as a 0-manifold, then y is a regular value of f, if and only if f is transverse to y. If the preimage of a point or submanifold is empty, Then the condition is satisfied vacuously.

Theorem 11.5. *If M, N are manifolds of dimension m, n respectively, and $Y \subset N$ is a smooth submanifold of codimension[6] k, then $f^{-1}(Y)$ is a smooth submanifold also with codimension k. For the special case of Y being a regular value, $f^{-1}(y)$ is a smooth submanifold of dimension $m - n$.*

The proof can be found here[2, p. 144]. Luckily for any smooth map between manifolds regular values are abundant.

Theorem 11.6 (Sard's Theorem)**.** *For any smooth map $f : M \to N$ between manifolds, the critical values are dense in N.*

The proof of Sard's theorem can be found here[2, p. 126]. The following lemma help us construct maps into $\mathbb{R}^k$ regular to $0 \in \mathbb{R}^k$.

Lemma 11.7. *Let $W \subset \mathbb{R}^m$ be an open subset, and $f : W \to \mathbb{R}^k$ be a smooth map with 0 as a regular value throughout a closed subset $X \in W$. Let K be a compact subset of W. There exists a smooth map $g : W \to \mathbb{R}^k$, such that 0 is a regular value throughout $X \cup K$, and g agrees with f outside a compact set. Given an $\epsilon > 0$ we can choose g, such that $|f(x) - g(x)| < \epsilon$ for any $x \in W$.*

[6] The codimension is difference between the dimension of the submanifold and the whole manifold.

Proof. We will construct g by slightly "pushing" f around K, such that the preimage of 0 becomes the preimage of a regular value close to 0, while leaving the rest of the function untouched.

First given a compact set K, let $K \subset U$ be a precompact neighborhood with $\overline{U} = K'$ and repeating the construction $K' \subset U'$ with U' a precompact neighborhood with $\overline{U'} = K''$.

we choose a partition of unity $\phi, (1-\phi)$ of $\{U', W - K'\}$ where U is a precompact neighborhood of K with $\overline{U} = K'$. From this we have $\phi : W \to \mathbb{R}$ with $\phi = 1$ in U and $\phi = 0$ outside of K''. Secondly from Sard's theorem, we can choose a regular value $y \in \mathbb{R}^k$ for f close to 0. We will later choose, how close it should be.

We can now define $g_y = f - y \cdot \phi$. For $k \in K$, $g_y(k) = 0$ if and only if $f(k) = y$. This show that $g_y^{-1}(0) \cap K = f^{-1}(y) \cap K$, and since ϕ is constant in U, $D(g_y)_x$ is surjective for $x \in K \cap g_y^{-1}(0)$, showing 0 is regular for g_y throughout K.

However we have modified our function on $X \cap K'$, so we have to make sure 0 is still regular for g_y in $X \cap K'$.

Given a small $0 < \delta$, let $B_\delta = d_{f^{-1}(0)}^{-1}([0, \delta))$ where $d_{f^{-1}(0)}$ is the distance function to $f^{-1}(0)$. If $f^{-1}(0) = \emptyset$, we let $B_\delta = \emptyset$. We can choose y small enough, such that $g_y^{-1}(0) \cap X \cap K' \subset B_\delta$ for any $0 < \delta$.

Indeed we have $X \cap K' \cap B_\delta^c$, where B_δ^c is the complement of B_δ, is a compact set where f is non-zero. Since it is compact, we can find ϵ such that $|f(w)| > \epsilon$ for $w \in X \cap K' \cap B_\delta^c$. But if $|y| < \epsilon$, we have $|f(w) - g_y(w)| < \epsilon$, so $g_y(w) \neq 0$ for $w \in X \cap K' \cap B_\delta^c$.

We have the continuous function $A : \mathbb{R}^k \times W \to \mathbb{R}$ that if $m \geq k$, assigns to $(y, x) \in \mathbb{R}^k \times W$

$$\max_{I, |I| = m-k} |\det(Dg_y)_{x,I}|,$$

where $(Dg_y)_{x,I}$ is the minor of $(Dg_y)_x$ obtained by removing columns corresponding to the set of indices I of length $m - k$. If $m < k$ then $A \equiv 0$. $(Dg_y)_x$ is surjective if and only if $A(y, x) \neq 0$.

Since $X \cap K' \cap f^{-1}(0)$ is compact, we have

$$\{0\} \times (X \cap K' \cap f^{-1}(0)) \hookrightarrow U \times V \hookrightarrow A^{-1}(0, \infty),$$

with U, V open. By compactness we can choose $\delta > 0$ such that $X \cap K' \cap B_\delta \in V$. Now choosing y with small enough absolute value, we both ensure that $y \in U$ and by above argument that $g^{-1}(0) \subset X \cap K' \cap B_\delta \subset V$. So given y satisfying this, for all $w \in g_y^{-1}(0)$ we have $A(y, w) \neq 0$ and so 0 is a regular value for g_y throughout $X \cup K$.

□

11.2 Thom's Theorem

We are now ready to describe the connection between homotopy groups of Thom spaces and cobordism groups. Given a smooth euclidean n-bundle ξ over a manifold B, since $T - t_0 \cong E(\xi)$, we can consider it as smooth on $T - t_0$.

Therefore if we have a continuous map $g : S^m \to T$, we can consider it smooth on $T - t_0$, if the map restricted to $g^{-1}(T - t_0)$ is smooth. From the Whitney approximation theorem, we can for any continuous map $g : S^m \to T$ find a homotopic map g', which is smooth on $g'^{-1}(T - t_0)$ and $g'^{-1}(T - t_0) = g^{-1}(T - t_0)$[7]. If it is possible to choose g' such that it is transverse to the zero-section B, we get a smooth submanifold $g'^{-1}(B) \subset S^m$ of dimension $m - n$.

The preimage $g'^{-1}(B)$ is however dependent on the choice of g' homotopic to g, but the following theorem shows that any map $g : S^m \to T$ is homotopic to such a g', and that $g'^{-1}(B)$ up to cobordism, does not depend on the choice of g'.

[7]This is possible since we can choose our approximation to be closer to g as it gets closer to t_0.

Theorem 11.8. *Using the notation as above, For any $g : S^m \to T$ there exist a homotopic map g', which is smooth on $T - t_0$ and transverse to B. The cobordism class of $g'^{-1}(B)$ is invariant under choice of homotopic map. Even further this gives a group homomorphism $\pi_m(T, t_0) \to \Omega^+_{m-n}$.*

Proof. First we will show that for any map $g : S^m \to T$, we can find a homotopic map satisfying the conditions. We approximate g by a map g_0 as above, which is smooth throughout $g_0^{-1}(T - t_0)$.

Since S^m is compact, we can choose an open covering of $g_0^{-1}(B)$ inside $g_0^{-1}(T-t_0)$ by finitely many open charts $W_1, W_2, ..., W_r$ such that $\pi \circ g_0(W_i) \subset U_i$, where U_i is open and the bundle is trivial over it. Furthermore choose compact sets $K_i \subset W_i$ such that the interior of $K_1 \cup K_2 ... \cup K_r$ contains $g_0^{-1}(B)$.

We will construct maps g_i inductively, by modifying the previous map g_{i-1} inside W_i. This is done so g_i is equal to g_{i-1} outside of a compact subset of W_i, transverse to B throughout $K_1 \cup K_2 ... \cup K_i$ and lastly $\pi \circ g_0 = \pi \circ g_i$ when restricted to $T - t_0$.

Assume we have already constructed g_{i-1}. Since $\pi \circ g_{i-1}(W_i) = \pi \circ g_0(W_i) \subset U_i$, given a trivialization diffeomorphism ϕ of U_i, we have

$$\phi \circ g_{i-1} : W_i \to (\pi \circ g_{i-1})(U_i) \times \mathbb{R}^n.$$

We let $\rho_i : W_i \to \mathbb{R}^n$ be the projection to the second factor. Note that g_{i-1} is transverse to B throughout the relatively closed subset $(K_1 \cup K_2 ... \cup K_{i-1}) \cap W_i$, if and only if ρ_i has 0 as a regular value throughout $(K_1 \cup K_2 ... \cup K_{i-1}) \cap W_i$.

Since W_i is diffeomorphic to a open subset of $\mathbb{R}^m$, by Lemma 11.7 we get a function ρ_i' which has 0 as a regular value throughout $(K_1 \cup K_2 ... \cup K_i) \cap W_i$, and is equal to ρ_i outside of a compact subset of W_i. Therefore we can replace $g_{i-1|W_i}$ by

$$\phi^{-1} \circ ((\pi \circ g_{i-1}) \times \rho_i'),$$

and leave everything else unchanged to get a map g_i satisfying the conditions. Using this inductively we have a map g_r homotopic to g, smooth on $g^{-1}(T - t_0)$ and having 0 as a regular value throughout $K_1 \cup K_2 ... \cup K_r$. However in the creation of g_r, we might have $g_r^{-1}(B) \cap (K_1 \cup K_2 ... \cup K_r)^c$ non-empty. This can be prevented by the following argument.

Note $g_0^{-1}(B)$ is inside the interior of $K_1 \cup K_2 ... \cup K_r$ so $S^m - (K_1 \cup K_2 ... \cup K_r)$ is contained in a compact set where $|g_0| > 0$. Using compactness there exists a $\epsilon > 0$ such that $|g_0| > \epsilon$ throughout $S^m - (K_1 \cup K_2 ... \cup K_r)$. Now by 11.7, we choose each g_i such that $|g_i - g_{i-1}| < \frac{\epsilon}{r}$. From this we get

$$|g_r| \geq |g_0| - |g_0 - g_r| > \epsilon - \epsilon = 0.$$

So $g_r^{-1}(0) \subset (K_1 \cup K_2 ... \cup K_r)$, and since 0 is regular in $K_1 \cup K_2 ... \cup K_r$, it is regular everywhere, showing the first part of the proof.

For the second part, assume that f, j both satisfy the assumptions and are homotopic. We will show that $f^{-1}(B)$ and $j^{-1}(B)$ are cobordant. By the Whitney approximation theorem there is a homotopy h_0 between f, j smooth on $h_0^{-1}(T - t_0)$. We can furthermore stretch it in the ends such that $h_0(x, t) = f(x)$ for $x \in [0, \frac{1}{3}]$ and $h_0(x, t) = t(x)$ for $x \in [\frac{2}{3}, 1]$.

We will now proceed as above, choosing open sets $W_1, W_2, ..., W_r$ covering $h_0^{-1}(B) \subset S^m \times [0, 1]$, and compact sets $K_i \subset W_i$ such that their interior also contains $h_0^{-1}(B)$. Let $K_i' = K_i \cap ([\frac{1}{3}, \frac{2}{3}] \times S^m)$ and $W_i' = W_i \cap (0, 1) \times S^m$. Now using Lemma 11.7 on W_i' as above, here with the relative closed subset

$$W_i' \cap \left(K_1 \cup ... \cup K_{i-1} \cup \left(\left[0, \frac{1}{3}\right] \times S^m\right) \cup \left(\left[\frac{2}{3}, 1\right] \times S^m\right)\right).$$

By an inductive argument as above, we get a homotopy h of f, j smooth on $h^{-1}(T - t_0)$ and transverse to B. It follows that $h^{-1}(B)$ is a smooth manifold with boundary $g^{-1}(B) \amalg t^{-1}(B)$, showing they are cobordant.

Lastly we need to show that this gives a group homomorphism $\pi_n(T, t_0) \to \Omega_{n-m}$. Since the map $[g] + [j] \in \pi_n(T, t_0)$ has representative given by g on the hemisphere above the equator and j on the lower hemisphere, the preimage will be $[g^{-1}(B) \amalg j^{-1}(B)] = [g^{-1}(B)] + [j^{-1}(B)]$ since they are separated by the equator which is disjoint from $h^{-1}(B)$, showing that the map is a group homomorphism. $\square$

If we assume ξ is oriented, we have an oriented version of this theorem.

Corollary 11.9. *If the bundle ξ is oriented, we get a group homomorphism $\pi_m(T, t_0) \to \Omega^+_{m-n}$.*

Proof. By the previous theorem, let $[g] \in \pi_m(T, t_0)$ with representative g which is smooth throughout $g^{-1}(T - t_0)$ and transverse to B. Recall that ξ is an euclidean bundle, so we have the normal bundle ν_m of $B \hookrightarrow T$. Note that $T - t_0$ is a tubular neighborhood of B inside T, so $E(\nu_m) \cong T - t_0 \subset E(\xi)$ with

$$\begin{tikzcd} E(v_m) \arrow[rr, "\cong"] \arrow[dr, "\pi_{\nu_m}"'] & & T - t_0 \arrow[dl, "\pi_{\xi|T-t_0}"] \\ & B & \end{tikzcd}$$

Composing with the diffeomorphism $T - t_0 \to E(\xi)$ by scaling vectors, we get a bundle map $\nu_m \to \xi$

$$\begin{tikzcd} E(v_m) \arrow[rr, "\cong"] \arrow[dr, "\pi_{\nu_m}"'] & & E(\xi) \arrow[dl, "\pi_{\xi|T-t_0}"] \\ & B & \end{tikzcd}$$

Showing that the two bundles are equivalent, and we give ν_m the orientation induced by orientation of ξ by the above map.

The pullback bundle on $g^{-1}(B)$ through g of ν_m is isomorphic to the normal bundle ν_n of $g^{-1}(B)$ inside S^m induced by a map $(Dg)^\perp$. We give ν_n the orientation induced by orientation of ν_m by the isomorphism. ν_n is then an oriented bundle inside the oriented bundle $\tau_{S^m|g^{-1}(B)}$ and since $\nu_n \oplus \tau_{g^{-1}(B)} = \tau_{S^m|g^{-1}(B)}$, $\tau_{g^{-1}(B)}$ must also be oriented, and so $g^{-1}(B)$ is an oriented manifold.
Using the same argument on a smooth homotopy between (g, j), we get an oriented cobordism $(h^{-1}(B), g^{-1}(B), f^{-1}(B))$, so the image of our map is well-defined up to oriented cobordism. As in the unoriented case $[g] + [j] \in \pi_n(T, t_0)$ get sent to $[g^{-1}(B) \amalg j^{-1}(B)] = [g^{-1}(B)] + [j^{-1}(B)]$ $\square$

We have now established that we have a homomorphism $\pi_n(T, t_0) \to \Omega^+_{n-m}$ for every oriented bundle ξ. The last piece of the puzzle is choosing the right bundle to get an interesting homomorphism.

Theorem 11.10. *The oriented tautological bundle $\tilde{\gamma}^k_p$ on $\tilde{G}_k(\mathbb{R}^{k+p})$ with $k, p \geq n$, gives a surjection*

$$\pi_{n+k}(T(\tilde{\gamma}^k_p)) \to \Omega^+_n.$$

Proof. Let M be a closed oriented manifold of dimension n. From the strong Whitney embedding theorem[2, p. 135] it can be embedded $\mathbb{R}^{n+k}$. We choose a tubular neighborhood U of M, which is diffeomorphic to $E(\nu^k)$. We define a bundle map $E(\nu^k) \to E(\tilde{\gamma}^k_n)$ by sending (x, v) to $v \in (T_xM)^\perp$, where $(T_xM)^\perp \hookrightarrow \mathbb{R}^{n+k}$ is the orthonormal space of the tangent space.
We have the following maps

$$U \cong E(\tilde{\gamma^k_n}) \hookrightarrow E(\tilde{\gamma}^k_p) \to T(\tilde{\gamma^k_p}),$$

where the last map is given by collapsing all vectors v with $|v| \geq 1$ to the basepoint t_0. This composition f is smooth on $f^{-1}(T - t_0)$ and transverse to the zero section, with $f^{-1}(\tilde{G}_k(\mathbb{R}^{k+p})) = M$.

If we take the one-point compactification of $\mathbb{R}^{n+k}$ with ∞, we get an embedding of U in S^{n+k}. We can extend the map f to $f' : S^{n+k} \to T(\tilde{\gamma^k_p})$ by sending $S^{n+k} - U$ to t_0. This map has the same properties as above, and it preserves basepoint as ∞ get sent to t_0, so we have therefore shown that the map $\pi_{n+k}(T(\tilde{\gamma}^k_p)) \to \Omega^+_n$ is surjective. $\square$

We have now obtained a surprising surjection between $\pi_{n+k}(T(\tilde{\gamma}^k_p))$ and Ω^+_n, but we do not know the isomorphism type of $\pi_{n+k}(T(\tilde{\gamma}^k_p))$ either. Up to torsion[8] however $\pi_{n+k}(T)$ and $H^{n+k}(T, t_0)$ are

[8] A homomorphism of abelian groups is an isomorphism up to torsion, if the kernel and cokernel is trivial.

isomorphic for all $n < k-1$[5, p. 208]. We can further compose with the Thom isomorphism $H^{n+k}(T, t_0) \to H^n(\tilde{G}_k(\mathbb{R}^{k+p}))$, to get $\pi_{n+k}(T) \cong H^n(\tilde{G}_k(\mathbb{R}^{k+p}))$ for $n < k-1$.
If we tensor all groups with $\mathbb{Q}$, we remove the torsion, and so get an isomorphism of groups. This is true since $\mathbb{Q}$ is flat so tensoring with $\mathbb{Q}$ is an exact functor, and if the kernel and cokernel are torsion groups, they vanish when we tensor with $\mathbb{Q}$. We know that for $p \geq n$, $H^n(\tilde{G}_k(\mathbb{R}^{k+p})) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H^n(\tilde{G}_k(\mathbb{R}^{k+p}); \mathbb{Q})$ from the universal coefficient theorem, so it is 0 if $4 \nmid n$, and if $n = 4r$ it is equal to the number of partitions of r into positive integers denoted $\kappa(r)$. Ω_n^+ is then a torsion group if $4 \nmid n$, and has at most rank $\kappa(r)$ when $n = 4r$.

However the products of complex projective spaces with total dimension n generate a rank $\kappa(r)$ subgroup of Ω_n^+ if $n = 4r$ as shown earlier. Therefore $\Omega_n^+ \otimes_{\mathbb{Z}} \mathbb{Q}$ is 0 if $4 \nmid n$, and has rank $\kappa(r)$ generated by all products of complex projective spaces with total dimension n if $n = 4r$.

We therefore get that $\Omega_*^+ \otimes_{\mathbb{Z}} \mathbb{Q}$ is the free polynomial $\mathbb{Q}$-algebra generated by the cobordism classes of $\mathbb{C}\mathrm{P}^{2n}$.

12 Multiplicative Sequence and Hirzebruchs Signature Theorem.

Given our calculation of $\Omega_+^* \otimes \mathbb{Q}$, we can classify a ring homomorphism $r : \Omega_+^* \otimes \mathbb{Q} \to \mathbb{Q}$ by the image of $[\mathbb{C}\mathrm{P}^{2n}]$ in $\mathbb{Q}$. We will describe such a ring homomorphism here. For a partition I of n we have the I-th Pontryagin number is a group homomorphism $\Omega_{4n}^+ \to \mathbb{Z}$. We can extend this to ring homomorphisms by multiplicative sequences.

Definition 12.1 (Multiplicative Sequences of $\mathbb{Q}$)**.** *Given a graded $\mathbb{Q}$-algebra A which is commutative*[9] *and a sequence of polynomials with coefficients in $\mathbb{Q}$, $\{K_i(x_1, ..., x_i)\}_{i \in \mathbb{N}}$ denoted K, with each K_i being homogeneous of degree i, we can evaluate an element $a \in A^{\Pi}$ on K by*

$$K(a) = 1 + K_1(a_1) + K_2(a_1, a_2) + ...$$

Then K is a multiplicative sequence if $K(ab) = K(a)K(b)$.

Let A be the strictly commutative graded $\mathbb{Q}$-algebra $\{H^{4i}(B, \mathbb{Q})\}$. Then the total Pontryagin class of a bundle ξ with $\mathbb{Q}$ coefficients $p(\xi) \in \{H^{4i}(B, \mathbb{Q})\}^{\Pi}$ is a formal sum, and we can evaluate such on a multiplicative sequence K to get $K(p(\xi)) \in \{H^{4i}(B, \mathbb{Q})\}^{\Pi}$.

Definition 12.2 (K-genus)**.** *For a multiplicative sequence K and an oriented closed $4n$-manifold M with fundamental class μ, the K-genus $K[M]$ is equal to*

$$\langle K_n(p_1(\tau), p_2(\tau), ..., p_n(\tau)), \mu \rangle \in \mathbb{Q},$$

and is equal to 0, if the dimension is not divisible by 4.

Lemma 12.3. *For a multiplicative sequence K, the K-genus is a ring homomorphism from Ω_*^+ to $\mathbb{Q}$.*

Proof. Since the Pontryagin numbers are a group homomorphism, we know the K-genus is additive on Ω_n^+, and so it is also additive on the direct sum Ω_*^+.
Assuming M, N are closed orientable manifolds of dimension $4m, 4n$ respectively. For the product of two manifolds $[M \times N]$ we have $\tau_{M \times N} = \pi_M^*(\tau_M) \oplus \pi_N^*(\tau_N)$, so $p(\tau_{M \times N}) = p(\tau_M) \otimes p(\tau_N)$ in $\mathbb{Q}$ coefficients. Since K is multiplicative we then have $K(p(\tau_{M \times N})) = K(p(\tau_M)) \times K(p(\tau_M))$. Furthermore the fundamental class $\mu_{M \times N}$ is the product $\mu_M \times \mu_N$. From this we get the calculation

$$\langle K(p(\tau_{M \times N})), \mu_{M \times N} \rangle = (-1)^{(4m)(4n)} \langle K(p(\tau_M)), \mu_M \rangle \langle K(p(\tau_N)), \mu_N \rangle$$

If one of the manifolds is not of dimension divisible of 4, then both $K[M]K[N]$ and $K[M \times N]$ is equal to 0. $\square$

[9]In this case we mean strictly commutative and not graded commutative.

The following result of Hirzebruch, will give many examples of multiplicative sequences K[5, p. 221].

Lemma 12.4. *Given a power series $f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + ...$ with coefficients in $\mathbb{Q}$, there is an unique associated multiplicative sequence K with coefficients in $\mathbb{Q}$, such that for $(1 + t) \in \mathbb{Q}[t]^{\Pi}$ we have*

$$K(1 + t) = f(t) \in \mathbb{Q}[t]^{\Pi}$$

For any graded symmetric $\mathbb{Q}$-algebra A with a_1 of degree 1, from the universal map $\mathbb{Q}[t] \to A$ sending t to a_1 we get

$$K(1 + a_1) = f(a_1) \in A^{\Pi}$$

12.1 Signature and Hirzebruch's Signature Theorem

We will now describe another ring homomorphism from Ω_*^+ to $\mathbb{Q}$. For any oriented closed topological manifold M of dimension $4n$, we get from Poincare duality that the cup product induce a commutative nonsingular pairing in the middle cohomology $H^{2n}(M;\mathbb{Q})$, since the middle degree is even and $H^{4n}(M;\mathbb{Q}) \cong \mathbb{Q}$ by sending the oriented generator to 1. The following lemma gives us an invariant of M.

Lemma 12.5. *Given a nonsingular symmetric pairing ϕ on a vector space V, we can construct a basis $\{x_1, ..., x_r, y_1, ..., y_s\}$, such that*

$$\phi(x_i, x_i) = 1$$
$$\phi(y_j, y_j) = -1$$
$$\phi(x_i, y_j) = \phi(y_j, y_{j'}) = \phi(x_i, x_{i'}) = 0$$

where $i \neq i'$ and $j \neq j'$. The number $r - s$ is invariant of choice of such basis, and is denoted the signature of ϕ.

The proof can be found here[4, p. 164]. From this we can define the signature.

Definition 12.6 (Signature of a closed oriented topological manifold)**.** *For a closed oriented topological $4n$-manifold M, the signature pairing induced by the cup product in the middle degree is the signature of M, denoted $\sigma(M)$*

We can show the signature is additive with the disjoint union and multiplicative with the cartesian product. Furthermore if M is oriented nul-cobordant then $\sigma(M) = 0$[4, p. 165]. From this we get $S : \Omega_*^+ \to \mathbb{Z}$ is a ring homomorphism, when restricted to smooth manifolds.

The following theorem gives another expression for signature.

Theorem 12.7 (Hirzebruch's Signature Theorem)**.** *The signature of smooth closed oriented manifold M is equal to $L[M]$, where L is the multiplicative sequence associated to the power series of the function $f(t) = \frac{\sqrt{t}}{\tanh \sqrt{t}}$*

$$1 + \frac{1}{3}t - \frac{1}{45}t^2 + ... + (-1)^{k-1} 2^{2^k} B_k t^k / (2k)! + ...$$

Where B_k is the k-th Bernoulli number. A description of the Bernoulli numbers can be found here [5, p. 281].

Proof. We can check that the two maps agree on the induced maps on $\Omega_*^+ \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}\mathrm{P}^{2i} | i \geq 1]$, since any torsion element of Ω_*^+ must be mapped to 0. To show the theorem, we then only have to check on the generators $\mathbb{C}\mathrm{P}^{2n}$.

We have the cohomology of $\mathbb{C}\mathrm{P}^{2n}$ with $\mathbb{Z}$ coefficients, is the polynomial ring $\mathbb{Z}[c_1(\gamma_{2n}^1)]/(c_1(\gamma_{2n}^1)^{n+1})$, and so by the ring inclusion $\mathbb{Z} \to \mathbb{Q}$ we get $\mathbb{Q}[a]/a^{2n+1}$ is the cohomology ring with $\mathbb{Q}$-coefficients. We have $a^n \cdot a^n = a^{2n}$ is the oriented generator of $H^{4n}(\mathbb{C}\mathrm{P}^{2n}, \mathbb{Q})$, so the signature of $\mathbb{C}\mathrm{P}^{2n}$ is 1.

The total Pontryagin class of $\mathbb{C}\mathrm{P}^{2n}$ is $(1 + a^2)^{2n+1}$, and so $L((1 + a^2)^{2n+1}) = L(1 + a^2)^{2n+1}$. Since the L-genus is characterised by $L(1 + x) = f(x)$ for $x \in H^4(\mathbb{C}\mathrm{P}^{2n};\mathbb{Q})$, we have $L((1 + a^2)^{2n+1}) = f(a^2)^{2n+1}$. Thus we have

$$\langle L(p), \mu \rangle = \langle f(a^2)^{2n+1}, \mu \rangle.$$

Since $\langle a^{2n}, \mu\rangle = 1$ and 0 for every other power, we have to determine the coefficient of a^{2n} in $f(a^2)^{2n+1}$. Since a does not have any relations in powers less than $2n+1$, we can replace a by a complex variable z to calculate the coefficient of a^{2n}. We therefore have to calculate the coefficient of z^{2k} in the Taylor expansion of $f(z^2)^{2n+1} = \left(\frac{z}{\tanh z}\right)^{2n+1}$. We now use Cauchy's integral formula for the n'th derivative

$$\frac{1}{2\pi i}\oint \frac{z^{2n+1}dz}{z^{2n+1}(\tanh z)^{2n+1}} = \frac{1}{2\pi i}\oint \frac{dz}{(\tanh z)^{2n+1}},$$

where $\oint$ is the integral around a loop of the origin. We now take the substitution $u = \tanh z$. We have $\frac{du}{dz} = 1-u^2$, and using the Taylor expansion $\frac{1}{1-u^2} = 1+u^2+u^4+...$ we get $dz = (1+u^2+u^4+...)du$. Since $\tanh z$ has the Taylor series

$$\tanh z = z - \frac{1}{3}z^2 + \cdots,$$

a loop around the origin with small enough absolute value, will get send to a loop around the origin, so doing the substitution we get a new integral around the origin.

$$\frac{1}{2\pi i}\oint \frac{(1+u^2+u^4+...)du}{u^{2n+1}} = \frac{1}{2\pi i}\oint (u^{-2n-1} + u^{-2n+1} + u^{-2n+3} + ...)du$$

By Cauchy's residue theorem we get the integral is equal to 1, and thus $L[\mathbb{C}\mathrm{P}^{2n}] = 1$. We have therefore shown L equals σ on the generators of Ω_*^+, and therefore they are equal for all closed smooth orientable manifolds. □

12.2 Consequences of the Hirzebruchs Signature Theorem

Hirzebruchs signature theorem has multiple consequences. First signature only depends on the cohomology ring and the choice of oriented generator of the top cohomology, so signature is invariant up to oriented homotopy equivalence, and so it holds for L-genus also.
Furthermore the signature is always an integer, and so the L-genus is also an integer. For the specific case of dimension 4, only the first Pontryagin class is non-zero, so $p_1[M]$ is divisible by 3. We will give a last example from Prof. Johannes Ebert, to see how the theorem can be used in calculations.

Corollary 12.8. *The manifold $A_{g,n} = \#_g(S^{4n+2} \times S^{4n+2})$ can only be an almost complex manifold if g is odd.*

Proof. For $n, g \geq 0$ consider the manifold $A_{g,n} = \#_g(S^{4n+2} \times S^{4n+2})$, which is the connected sum of g copies of $S^{4n+2} \times S^{4n+2}$. For $g = 0$ we define it to be $S^{4(2n+1)}$.
First we will compute the signature of $A_{g,n}$, but since the signature is multiplicative under products and additive under connected sum, we get $\sigma(A_{g,n}) = 0$, since $\sigma(S^m) = 0$ for any sphere. We will now estimate $L[A_{g,n}]$. Assuming $A_{g,n}$ has a complex structure agreeing with the standard orientation on the tangent bundle, we have

$$1 - p_1 + p_2 - ... + p_{4n+2} = (1 + c_1 + c_2 + ... + c_{4n+2})(1 - c_1 + c_2 + ... + c_{4n+2})$$

The cohomology of $A_{g,n}$ is given by

$$H^m(A_{g,n}) = \begin{cases} \mathbb{Z} & m = 0 \\ \mathbb{Z}^{2g} & m = 4n+2 \\ \mathbb{Z} & m = 8n+4 \\ 0 & else \end{cases}$$

So all Chern classes must vanish except for c_{2n+1} and c_{4n+2} giving us

$$1 - p_1 + p_2 - ... + p_{4n+2} = (1 + c_{2n+1} + c_{4n+2})(1 - c_{2n+1} + c_{4n+2}) = 1 + 2c_{4n+2} - c_{2n+1}^2$$

So the only non-vanishing Pontryagin class is $p_{2n+1} = c_{2n+1}^2 - 2c_{4n+2}$. It follows

$$L[A_{g,n}] = q \cdot (c_{2n+1}^2[A_{g,n}] - 2c_{4n+2}[A_{g,n}]),$$

where q is the coefficient of p_n in L_n, which is non-zero[1]. From Hirzebruch signature theorem we get $L[A_{g,n}] = \sigma(A_{g,n}) = 0$ so $c_{2n+1}^2[A_{g,n}] = 2c_{4n+2}[A_{g,n}]$. Since c_{4n+2} is the Euler class we get

$2c_{4n+2}[A_{g,n}] = 2\chi(A_{g,n}) = 4(g+1)$ where $\chi(A_{g,n})$ is the Euler characteristic which can be computed from cohomology.

We have under the coefficient change $\mathbb{Z} \to \mathbb{Z}/2$ the Chern class c_{2n+1} get send to w_{4n+2}. We will now show $w_{4n+2} = 0$.
We can consider $A_{g,n}$ as a cellular complex with the 0-skeleton a point, the $4n+2$ skeleton a wedge of $2g$ copies of S^{4n+2} and a $(8n+4)$-cell attached along the spheres. The $4n+3$ skeleton $(A_{g,n})_{4n+3}$ of $A_{g,n}$ is then also the wedge of $2g$ copies of S^{4n+2}, and since the inclusion of the $4n+3$ skeleton induces isomorphisms on cohomology in $4n+2$ we have

$$w_{4n+2}(\tau_{A_{g,n}}) = w_{4n+2}((\tau_{A_{g,n}})_{|(A_{g,n})_{4n+3}})$$

Since $(A_{g,n})_{4n+3}$ is a wedge of copies of S^{4n+2}, the cohomology class is the sum of the class restricted to each sphere in the wedge, so we will calculate $w_{4n+2}((\tau_{A_{g,n}})_{|S^{4n+2}})$ for a sphere in the wedge $S^{4n+2} \hookrightarrow (A_{g,n})_{4n+3} \hookrightarrow A_{g,n}$.
By thickening the S^{4n+2} inside $S^{4n+2} \times I^{4n+2} \hookrightarrow S^{4n+2} \times S^{4n+2} \hookrightarrow A_{g,n}$, we see that the normal bundle of S^{4n+2} is trivial so

$$(\tau_{A_{g,n}})_{|S^{4n+2}} \cong \tau_{S^{4n+2}} \oplus \epsilon^{4n+2} \cong \epsilon^{8n+4}$$

Since $\tau_{S^{4n+2}} \oplus \epsilon \cong \epsilon^{4n+3}$, so it follows $w_{4n+2} = 0$ when restricted to any sphere in the wedge, and so $(\tau_{A_{g,n}})_{|(A_{g,n})_{4n+3}}) = 0$ which then implies also $w_{4n+2}(\tau_{A_{g,n}}) = 0$.

Since $w_{4n+2} = 0$, we must have $c_{2n+1} = 2x$ for some $x \in H^{4n+2}(A_{g,n})$. Choose a basis $(\alpha_1, \beta_1, ..., \alpha_g, \beta_g)$ for $H^{4n+2}(A_{g,n})$ with

$$\alpha_i \beta_i = c$$

$$\alpha_i \beta_j = \alpha_i \alpha_{i'} = \beta_j \beta_{j'} = 0$$

where $c = H^{8n+4}(A_{g,n})$ is the oriented generator and $i \neq j$. Then we have

$$x = a_1\alpha_1 + b_1\beta_1 + ... + a_1\alpha_1 + b_g\beta_g$$

where $a_i, b_i \in \mathbb{Z}$ and

$$c_{2n+1}[A_{g,n}] = 4x^2[A_{g,n}] = 8(a_1b_1 + ... + a_gb_g)c[A_{g,n}] = 8(a_1b_1 + ... + a_gb_g).$$

It follows

$$8(a_1b_1 + ... + a_gb_g) = 4(g+1)$$

So g must be odd. □

The manifold $A_{1,0}$ can be given a almost complex structure, since it it the product of $S^2 \times S^2$, and S^2 can be given an almost complex structure since $S^2 \cong \mathbb{C}\mathrm{P}^1$.

References

[1] A. Berglund and J. Bergström. Hirzebruch L-polynomials and multiple zeta values. *Mathematische Annalen*, 372:125–137, January 2018.

[2] J. Lee. *Introduction to Smooth Manifolds.* Springer, second edition, 2012.

[3] Y. Matsumoto. *An Introduction to Morse Theory.* Iwanami Shoten, 1997.

[4] J. P. May. *A Concise Course in Algebraic Topology.* University of Chicago Press, 1999.

[5] J. W. Milnor and J. D. Stasheff. *Characteristic Classes.* Princeton University Press, 1974.

[6] C. Rourke. A new proof that Ω_3^+ is zero. *J. London Math. Soc*, 31(Part 2):373–376, April 1985.